Weinberg Bounds over Nonspherical Graphs

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Abstract: Let $\text{Aut}(G)$ and $E(G)$ denote the automorphism group and the edge set of a graph G, respectively. Weinberg's Theorem states that 4 is a constant sharp upper bound on the ratio $|\text{Aut}(G)|/|E(G)|$ over planar (or spherical) 3-connected graphs G. We have obtained various analogues of this theorem for nonspherical graphs, introducing two Weinberg-type bounds for an arbitrary closed surface Σ, namely:

$$W_P(\Sigma) \text{ and } W_T(\Sigma) \overset{\text{def}}{=} \sup_G |\text{Aut}(G)|/|E(G)|,$$

where supremum is taken over the polyhedral graphs G with respect to Σ for $W_P(\Sigma)$ and over the graphs G triangulating Σ for $W_T(\Sigma)$. We have proved that Weinberg bounds are finite for any surface; in particular: $W_P = W_T = 48$ for the projective plane, and $W_T = 240$ for the torus. We have also proved that the original Weinberg bound of 4 holds over the graphs G triangulating the projective plane with at least 8 vertices and, in general, for the graphs of sufficiently large order triangulating a fixed closed surface Σ.

Keywords: *automorphism group; 3-connected graph; surface; triangulation*

1. INTRODUCTION

The term "surface" always means a compact 2-manifold without boundary. By Σ_g, $g \geq 0$ [respectively, $\tilde{\Sigma}_k$, $k > 0$], we denote the 2-sphere Σ_0 fitted with g handles [k crosscaps]. The term "graph" disallows loops and multiple edges. Let $\Psi : G \to \Sigma$ be an embedding of a graph G in a fixed surface Σ. Graph G is called the *graph of embedding* Ψ and is denoted by $G(\Psi)$. The *faces* of Ψ are the closures of the connected components of $\Sigma \setminus \Psi(G)$. We say Ψ is an embedding with *representativity* ρ, if ρ is maximum such that every nontrivial (i.e., nonnull-homotopic) closed curve in Σ intersects $\Psi(G)$ at least ρ times; also, Ψ is *n-representative* on Σ, if $\rho \geq n$. We shall restrict the type of embeddings considered to *polyhedral embeddings*, more precisely, embeddings of 3-connected graphs for $\Sigma = \Sigma_0$, and 3-representative embeddings of 3-connected graphs for $\Sigma \neq \Sigma_0$. In a polyhedral embedding $G \to \Sigma$, each face is bounded by a (simple) cycle of G, and the subgraph of G bounding the faces incident with any vertex is a wheel with at least 3 spokes and a possibly subdivided rim; see [14]. Especially, a *triangulation* is an embedding $T : G \to \Sigma$ with every face bounded by a 3-cycle of G, i.e., a cycle of length 3. We say that a 3-connected graph G is a *polyhedral graph* with respect to Σ, if some embedding $G \to \Sigma$ is 3-representative and no such embedding is 2-representative. The class of polyhedral graphs with respect to Σ is denoted by $\mathrm{PG}(\Sigma)$. The graphs that triangulate Σ form a subclass in $\mathrm{PG}(\Sigma)$, denoted by $\mathrm{TG}(\Sigma)$. By $V(\cdot)$, $E(\cdot)$ and $F(\cdot)$, we denote the sets of vertices, edges and faces, respectively. We treat embeddings combinatorially rather than topologically, assuming $\Psi : G \to \Sigma$ is well-defined by its graph $G(\Psi) = \{V(G), E(G)\}$ together with the face set $F(\Psi)$. Combinatorially, $F(\Psi)$ is a collection of the cycles of G bounding the faces of Ψ. The automorphism group of graph G is denoted by $\mathrm{Aut}(G)$. The following is a celebrated theorem of Weinberg.

Theorem 1 (Weinberg [16]). *For every planar (or spherical)* 3*-connected graph* G, *we have*

$$\frac{|\mathrm{Aut}(G)|}{|E(G)|} \leq 4 \tag{1}$$

Furthermore, equality holds if any only if G *is the* 1*-skeleton of one of the five Platonic solids.* ∎

The purpose of this article is to develop analogues of this theorem for the classes $\mathrm{PG}(\Sigma)$ and $\mathrm{TG}(\Sigma)$ of graphs in an arbitrary surface Σ, giving a useful reinterpretation of the type of results (e.g., [3–8, 10, 14, 15]) counting the number of different embeddings of a graph in fixed surface.

Definition 1. *The Weinberg bound* $W_P(\Sigma)$ *for a fixed surface* Σ *is defined by*

$$W_P(\Sigma) \overset{\mathrm{def}}{=} \sup_{G \in \mathrm{PG}(\Sigma)} \frac{|\mathrm{Aut}(G)|}{|E(G)|}.$$

The Weinberg bound $W_T(\Sigma)$ *is defined by the same equation, replacing* $\mathrm{PG}(\Sigma)$ *by* $\mathrm{TG}(\Sigma)$.

Theorem 2. *The Weinberg bounds* $W_P(\Sigma)$ *and* $W_T(\Sigma)$ *are finite for any surface* Σ.

The proof of this theorem is given in Section 4.

Theorem 1 (Weinberg's Theorem), in fact, states that $W_P(\Sigma_0) = W_T(\Sigma_0) = 4$. As for the nonspherical surfaces, the authors have previously established [4] that $W_T(\tilde{\Sigma}_2) = 16$ for the Klein bottle $\tilde{\Sigma}_2$. The following is a result in the sequence.

Theorem 3. $W_P(\tilde{\Sigma}_1) = W_T(\tilde{\Sigma}_1) = 48$ *for the projective plane* $\tilde{\Sigma}_1$, *and* $W_T(\Sigma_1) = 240$ *for the torus* Σ_1. *Furthermore, the former two bounds are attained if and only if* $G = K_6$, *the complete graph of order* 6, *and the latter is attained if and only if* $G = K_7$, *the complete graph of order* 7.

The inclusion $\mathrm{TG}(\Sigma) \subset \mathrm{PG}(\Sigma)$ implies the following inequality:

$$W_T(\Sigma) \le W_P(\Sigma), \tag{2}$$

for each surface Σ. One might expect that $W_P(\Sigma) = W_T(\Sigma)$ for any surface Σ. This is indeed the case for $\Sigma = \Sigma_0$ and $\Sigma = \tilde{\Sigma}_1$, by Theorems 1 and 3, respectively; for other surfaces, this is our conjecture. The proof of Theorem 3 is postponed until Section 4. In Sections 2 and 3, we develop a group-theoretic approach to the study of the automorphism group of a given graph, using its embeddings in a suitable surface. In Section 4, we establish (Lemma 9) that the phenomenon of flexibility, described in Section 2, occurs thanks to only a finite number if "flexible" faces. This result gives a new insight into the study of graphs in higher-genus surfaces; in particular, it gives rise to the following very general principle (which seems to be reasonable if one restricts the sort of theorems one is looking at).

Conjecture 1 (polyhedrality principle). If a theorem holds for any spherical 3-connected graph, then it also holds for any polyhedral graph, with respect to a fixed surface, subject to at most finitely many exceptions.

The following is a particular case of the polyhedrality principle, which is in the focus of our attention in this article.

Conjecture 2. The Weinberg bound (1) holds over almost all polyhedral graphs G, more precisely, for each surface Σ there exists an integral constant $C(\Sigma)$ such that the Weinberg bound (1) holds for any graph $G \in \mathrm{PG}(\Sigma)$ of order at least $C(\Sigma)$.

In Section 4, we affirmatively prove this conjecture restricted to $\mathrm{TG}(\Sigma)$, establishing (not constructively) the following theorem.

Theorem 4. *For each surface* Σ, *there is an integral constant* $C(\Sigma)$ *such that the Weinberg bound* (1) *holds for any graph* $G \in \mathrm{TG}(\Sigma)$ *as long as* $|V(G)| \ge C(\Sigma)$.

Actually, a stronger statement is proved, namely that $|\mathrm{Aut}(G)|$ is bounded above by a constant depending only on the surface, but not on the graph G. Also in Section 4, we obtain the result of Theorem 4 constructively for the projective plane.

Theorem 5. *The Weinberg bound* (1) *holds for any graph* G *triangulating the projective plane with at least eight vertices, but is never attained by any such graph.*

Therefore, on the practical side, Conjecture 2, if proved to be true, would not yet provide a fully satisfactory answer, since the bound of 4 may be only attained by the graphs of regular maps, with small graphs excluded by the condition on the number of edges. Therefore, because of the scarcity of regular maps, noticed by Nakamoto and Negami [9], it is sensible to study "Weinberg limit superiors", which is addressed in Section 5.

2. FLEXIBILITY

Let Ψ, $\Psi' : G \to \Sigma$ be two embeddings. They are regarded as (*combinatorially*) *equivalent*, provided that $F(\Psi) = F(\Psi')$, i.e., any cycle of G bounds a face either in both Ψ and Ψ' (not necessarily with the same orientation) or in neither; otherwise Ψ and Ψ' are regarded as *distinct*. An *isomorphism* $\pi : \Psi \to \Psi'$ is an automorphism of G such that $\upsilon_1\upsilon_2\cdots\upsilon_l \in F(\Psi)$ if and only if $\pi(\upsilon_1)\pi(\upsilon_2)\cdots\pi(\upsilon_l) \in F(\Psi')$; faces are designated by listing the incident vertices following the natural circular order of their boundaries (regardless of the orientation). Especially, an isomorphism $\Psi \to \Psi$ is called an *automorphism* of Ψ. Clearly, the automorphism group $\mathrm{Aut}(\Psi)$ of Ψ is a subgroup of $\mathrm{Aut}(G)$.

Note 1. Distinct [respectively, nonisomorphic] embeddings are distinguishable in the vertex-labeled [vertex-unlabeled] sense. Distinct embeddings may be isomorphic or, in other words, identical, if the labels are neglected.

An important theorem of Whitney [17] states that an embedding Ψ of a 3-connected planar graph in the plane (or the 2-sphere Σ_0) is combinatorially unique, i.e., the face set $F(\Psi)$ is uniquely determined by the graph $G(\Psi)$. In fact, Whitney's Theorem implies Theorem 1, since it guarantees the identity $\mathrm{Aut}(\Psi) = \mathrm{Aut}(G(\Psi))$ whenever $G(\Psi)$ satisfies the hypothesis of Theorem 1. We shall give out proof of Theorem 1 shortly, but first we introduce one more important concept.

Definition 2. A flag *of an embedding* Ψ *is an incident vertex-edge-face triple, i.e., a triple of the form* $\{u, u\upsilon, u\upsilon w_1\cdots w_l\}$.

Clearly, every edge of a polyhedral embedding gives rise to exactly four flags, and one can derive the Weinberg bound (1) for $G(\Psi)$ from the following lemma, whose proof is obvious.

Lemma 1. *Let* Ψ *be an embedding* $G \to \Sigma$. *There are exactly* $4|E(G)|$ *flags in* Ψ. *Furthermore, each automorphism of* Ψ *is uniquely determined by its effect on any one flag of* Ψ. ■

Note 2. In Lemma 1, and throughout the article, we implicitly assume that all our embeddings are polyhedral. Otherwise Definition 2 does not work as one expects when the embeddings of a graph has an edge with both sides incident with the same face, for the trivial action on a flag with such an edge might correspond to two distinct automorphisms, one of which is the identity and the other switches the sides of the edge, fixing it.

By Lemma 1,

$$|\mathrm{Aut}(\Psi)| \le 4|E(G)|. \tag{3}$$

Furthermore, when this upper bound is attained, $\mathrm{Aut}(\Psi)$ acts transitively on the flag set, in which event Ψ is called a *regular map*. Clearly, the faces of a regular map are bounded by the same number of edges and its vertices are incident with the same number of edges. This is possible in the 2-sphere if and only if Ψ is an embedding of the 1-skeleton of a Platonic solid. We have thus proved Theorem 1.

The reader may note that the proof given applies to any graph $G \in \mathrm{PG}(\Sigma)$ uniquely embeddable into Σ. Consequently, we have a simple sufficient condition for the nonuniqueness of embedding $G \to \Sigma$, namely: *Inequality* (1) *be violated*; for example, the complete graph K_P is nonuniquely embeddable (in a fixed surface) unless $p \le 4$. In this section, we continue the development of our approach to graph re-embedding theory, which was begun in [5–8].

For $\Sigma \ne \Sigma_0$, Whitney's Theorem fails and there may exist two or more distinct embeddings $G(\Psi) \to \Sigma$ with the graph $G(\Psi)$ of a given embedding Ψ in Σ. For example, the embeddings of K_P mentioned in the preceding paragraph are nonunique. In particular, we see in the next section (Lemma 2) that there are 12 distinct triangulations $K_6 \to \tilde{\Sigma}_1$.

Let Ψ be a fixed embedding of a fixed graph $G = G(\Psi)$ in a fixed surface $\Sigma \ne \Sigma_0$. Denote by $\{\mathrm{FLEX}_i(\Psi)\}_{i=0}^{\mathrm{flex}(\Psi,\Sigma)}$ the set of the *flexes* of embedding Ψ in surface Σ, i.e., the set of pairwise distinct (labeled) embeddings $G \to \Sigma$. As matter of notation, we assume that $\mathrm{FLEX}_0(\Psi)$ is embedding Ψ itself, regarding it as a *trivial flex*. The *flexibility* $\mathrm{flex}(\Psi,\Sigma)$ is defined to be the number of nontrivial flexes of Ψ in Σ. Therefore,

$$\left|\{\mathrm{FLEX}_i(\Psi)\}\right| = 1 + \mathrm{flex}(\Psi,\Sigma).$$

An embedding Ψ with flexibility at least k is called *k-flexible* (in Σ). A *flexible* embedding is one that is 1-flexible. A *nonflexible*, or *rigid*, embedding is one with flexibility 0.

Some of the flexes of Ψ may be isomorphic to each other, but not necessarily. For example, the twelve distinct embeddings $K_6 \to \tilde{\Sigma}_1$ are isomorphic triangulations (Lemma 2). Examples of nonisomorphic triangulations of $\tilde{\Sigma}_1$ with the same graph may be found in [5, 6, 8]. The group $\mathrm{Aut}(G)$ naturally acts on the flex set of Ψ, more precisely, as follows:

$$F(\alpha \mathrm{FLEX}_i(\Psi)) \stackrel{\mathrm{def}}{=} \{\alpha(u)\alpha(\upsilon)\alpha(w_1)\cdots\alpha(w_l) : u\upsilon w_1 \cdots w_l \in F(\mathrm{FLEX}_i(\Psi))\},$$

for $\alpha \in \mathrm{Aut}(G)$. Observe that $\mathrm{FLEX}_i(\Psi)$ and $\mathrm{FLEX}_j(\Psi)$ are isomorphic if and only if there is $\alpha \in \mathrm{Aut}(G)$ such that $\alpha \mathrm{FLEX}_i(\Psi) = \mathrm{FLEX}_j(\Psi)$. The set $\{\mathrm{FLEX}_i(\Psi)\}$ thus breaks into, say N, isomorphism classes ("orbits"). Pick a representative, Φ_n, in the nth class. Note that $\mathrm{Aut}(\Phi_n)$ is a subgroup of $\mathrm{Aut}(G)$ that coincides with the stabilizer of Φ_n. The size of nth class is given by the index $[\mathrm{Aut}(G) : \mathrm{Aut}(\Phi_n)]$, which indicates the number of distinct embeddings $G \to \Sigma$ isomorphic to Φ_n. The following is the well-known orbit-stabilizer formula for decomposition into orbits:

$$1+\mathrm{flex}(\Psi,\Sigma)=\sum_{n=1}^{N}[\mathrm{Aut}(G):\mathrm{Aut}(\Phi_n)]=\sum_{n=1}^{N}\frac{|\mathrm{Aut}(G)|}{|\mathrm{Aut}(\Phi_n)|} \tag{4}$$

In these two sums, the terms with the same index n are pairwise equal, so that

$$|\mathrm{Aut}(G)|=[\mathrm{Aut}(G):\mathrm{Aut}(\Psi)]\cdot|\mathrm{Aut}(\Psi)|\le(1+\mathrm{flex}(\Psi,\Sigma))\cdot|\mathrm{Aut}(\Psi)|. \tag{5}$$

Hence, by Inequality (3), we have

$$|\mathrm{Aut}(G)|\le 4(1+\mathrm{flex}(\Psi,\Sigma))\cdot|E(G)|. \tag{6}$$

3. TRIANGULATIONS

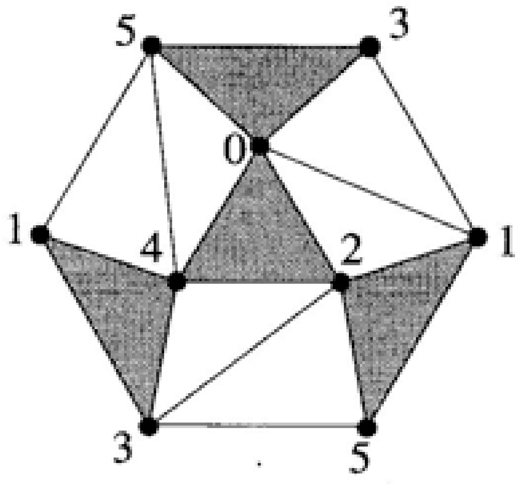
A well-known consequence of Euler's equation for surface Σ is that, if an embedding Ψ is a triangulation of Σ, then any embedding $G(\Psi)\to\Sigma$ is a triangulation of Σ.

We depict the projective plane $\tilde{\Sigma}_1$ as a regular hexagon with antipodal points on the boundary treated as identical. The Fig. 1(a) [or 1(c)] presents a triangulation $K_6\to\tilde{\Sigma}_1$, with the vertices of K_6 labeled by 0–5.

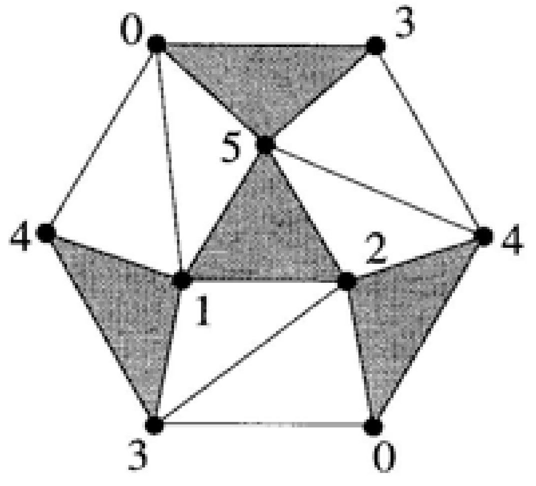
In this section, we include the proofs of some known results, which help clear up the phenomenon of flexibility.

Lemma 2 (Negami [12], Lawrencenko [7, 8], Vitray and Robertson [14, 15]). *The complete graph K_6 has exactly twelve distinct labeled embeddings in the projective plane $\tilde{\Sigma}_1$, all of which are isomorphic triangulations.* ■

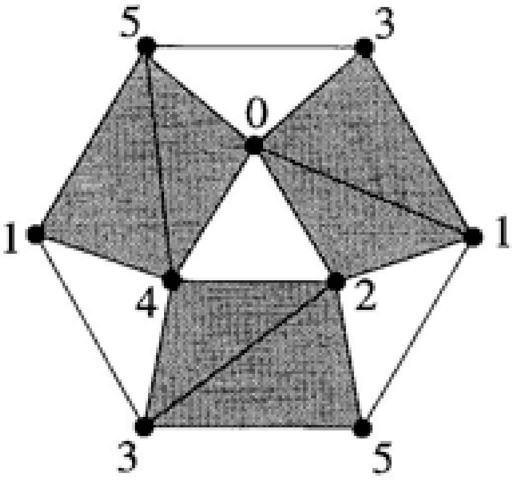
Proof. On one hand, by Euler's equation, every embedding $K_6\to\tilde{\Sigma}_1$ is a triangulation with 10 faces. On the other hand, K_6 consists of 10 pairs of disjoint 3-cycles. By an obvious topological argument, in each of the pairs, one and only one 3-cycle is bounding. As matter of notation, assume that 3-cycle (0, 2, 4, 0) bounds a face, whence (1, 3, 5, 1) does not. Cutting the projective plane open around this nonbounding cycle results in a hexagon with vertices 1, 3 and 5 on its boundary, and face 024 strictly inside, as in Fig. 1(a). First, there are three choices for a second face meeting the edge 02 (i.e., the edge between the vertices 0 and 2), namely: 021, 023 and 025. Second, for each of these choices, there are two choices for a second face meeting the edge 04; for instance, for the first choice of 021, these two choices are 045 and 043. Once the second choice is made, the remaining faces are determined uniquely; for instance, for the first choice of 021 and the second choice of 045, the unique embedding $K_6\to\tilde{\Sigma}_1$ is depicted in Fig. 1(a). In this fashion, we can construct $3\cdot 2$ distinct triangulations, and 6 more are obtained by interchanging the roles of 3-cycles (0, 2, 4, 0) and (1, 3, 5, 1).



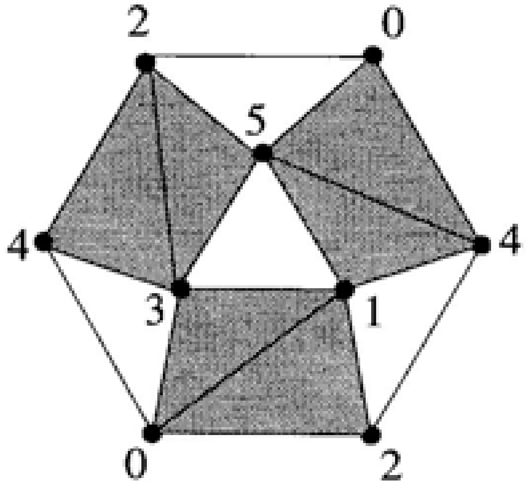
a: IT1=FLEX0(IT1)



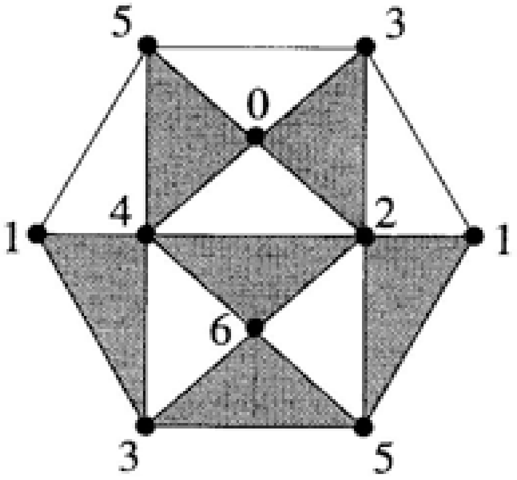
b: FLEX1(IT1)



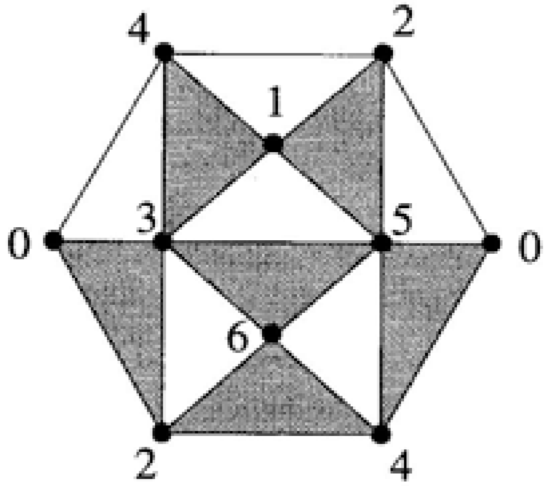
c: IT1=FLEX0(IT1)



d: FLEX2(IT1)



e: IT2=FLEX0(IT2)



f: FLEX1(IT2)

FIGURE 1. Triangulations of the projective plane.

Let $T:G \to \Sigma$ be a triangulation of a fixed surface Σ, not the 2-sphere Σ_0. The operation of *shrinking* an edge $\upsilon_1\upsilon_2$ is denoted by $\mathrm{sh}\rangle\upsilon_1\upsilon_2\langle$ and consists of collapsing the edge to a single vertex, υ, and the two incident faces, $\upsilon_1\upsilon_2 u$ and $\upsilon_1\upsilon_2 w$, to two edges, υu and υw, respectively. The inverse of this operation is called the *splitting*, $\mathrm{sp}\langle v,u,w\rangle$, of the *corner* $\langle u, \upsilon, w\rangle$, i.e., the pair of edges $\{\upsilon u, \upsilon w\}$.

Note 3. Every edge of a triangulation T occurs in the boundaries of exactly two faces.

If an edge occurs in more than two 3-cycles of G, it is called *unshrinkable*. If one insisted on shrinking an unshrinkable edge xy, this process would result in a multigraph. For, under the shrinking, the nonfacial 3-cycle (x, y, z, x) determined by the edge xy and some vertex z would transform to a pair of multiple edges joining vertex $x = y$ with vertex z.

We say T is an *irreducible triangulation* (of $\Sigma \neq \Sigma_0$) if each edge of T is unshrinkable; none of its edges can be shrunk further. Clearly, the whole family of triangulations of Σ can be obtained from the irreducible ones by repeatedly applying the operation of splitting. We use the complete list, up to isomorphisms, of (two) irreducible triangulations of the projective plane $\tilde{\Sigma}_1$ identified by Barnette [1]. A complete list of (twenty-one) irreducible triangulations of the torus Σ_1 is identified by Lawrencenko [5]. The two irreducible triangulations of $\tilde{\Sigma}_1$ are presented, respectively, in Fig. 1(a), denoted by IT_1 and Fig. 1(e), denoted by IT_2.

Lemma 3 (Barnett and Edelson [2]). *There are at most finitely many irreducible triangulations of any surface.* ■

All flexes of IT_1 and IT_2 in the projective plane are presented in Table 1 (from [8]). The trivial flexes $\mathrm{FLEX}_0(IT_1) = IT_1$ and $\mathrm{FLEX}_0(IT_2) = IT_2$ are depicted in Figs. 1(a) [or 1(c)] and 1(e), respectively. Every row in Table 1 is a permutation of the first row. To obtain a picture of $\mathrm{FLEX}_i(IT_1)$, merely replace the labels in Fig. 1(a) as the ith permutation prescribes; and similarly for $\mathrm{FLEX}_i(IT_2)$. For instance, to obtain $\mathrm{FLEX}_1(IT_1)$, replace the labels 0, 1, 2, 3, 4, 5 in Fig. 1(a) with 5, 4, 2, 3, 1, 0, respectively; see Fig. 1(b). Similarly, $\mathrm{FLEX}_2(IT_1)$ and $\mathrm{FLEX}_1(IT_2)$ are shown in Figs. 1(d) and 1(f), respectively.

TABLE 1. Flexes of the Irreducible Triangulations of the Projective Plane.

$\mathrm{FLEX}_0(IT_1)$:	0	1	2	3	4	5	$\mathrm{FLEX}_0(IT_2)$:	0	1	2	3	4	5	6
$\mathrm{FLEX}_1(IT_1)$:	5	4	2	3	1	0	$\mathrm{FLEX}_1(IT_2)$:	1	0	5	2	3	4	6
$\mathrm{FLEX}_2(IT_1)$:	5	4	1	0	3	2	$\mathrm{FLEX}_2(IT_2)$:	1	0	2	3	4	5	6
$\mathrm{FLEX}_3(IT_1)$:	1	5	2	3	4	0	$\mathrm{FLEX}_3(IT_2)$:	1	6	2	3	4	5	0
$\mathrm{FLEX}_4(IT_1)$:	5	0	2	3	4	1	$\mathrm{FLEX}_4(IT_2)$:	0	6	2	3	4	5	1
$\mathrm{FLEX}_5(IT_1)$:	5	1	2	3	4	0	$\mathrm{FLEX}_5(IT_2)$:	6	1	2	3	4	5	0
$\mathrm{FLEX}_6(IT_1)$:	0	4	2	3	1	5								
$\mathrm{FLEX}_7(IT_1)$:	0	5	2	3	4	1								
$\mathrm{FLEX}_8(IT_1)$:	4	5	2	3	1	0								
$\mathrm{FLEX}_9(IT_1)$:	4	0	2	3	1	5								
$\mathrm{FLEX}_{10}(IT_1)$:	1	4	2	3	5	0								
$\mathrm{FLEX}_{11}(\mathrm{IT}_1)$:	4	1	2	3	5	0								

Note 4. To understand the proof of Lemma 12 in the next section, it is helpful to draw pictures of the twelve flexes of IT_1 and the six flexes of IT_2.

Definition 3. *A face of a triangulation* T *of a surface* Σ *is called* rigid, *with respect to* Σ, *if it is a face of each flex of* T, *and is called* flexible *otherwise.*

Note 5. Each boundary edge of a flexible face necessarily occurs in more than two 3-cycles of $G(T)$; recall Note 3.

Clearly, a triangulation is rigid if and only if every its face is rigid. The following is a useful observation, whose proof is obvious.

Lemma 4. *The two new faces produced by a splitting are always rigid, and splitting preserves the rigidity of a face. Hence, the number of flexible faces cannot increase by splitting.* ■

The flex set $\{\mathrm{FLEX}_i(T)\}$ evolves under splittings of T; some of the flexes survive and some are destroyed. Let $T' = \mathrm{sp}\langle v,u,w\rangle\,(T)$ and let υ_1, υ_2 denote the two images of vertex υ under the splitting.

Clearly, for each j, there is a unique i such that the following equality holds:

$$\mathrm{sh}\rangle\upsilon_1\upsilon_2\langle\left(\mathrm{FLEX}_j(T')\right) = \mathrm{FLEX}_i(T). \tag{7}$$

We say that $\mathrm{FLEX}_i(T)$ *survives* under the splitting $sp\langle u,\upsilon,w\rangle$ provided that Eq. (7) holds for some j. Two corners $\langle u,\upsilon,w\rangle$ and $\langle x,\upsilon,y\rangle$ are said to *cross* each other (at vertex υ) in triangulation T, if there is a homeomorphism of $\mathrm{star}(\upsilon,T)$ onto the unit disk in the complex plane such that the image of $\langle u,\upsilon,w\rangle$ follows the real axis and the image of $\langle x,\upsilon,y\rangle$ follows the imaginary axis.

Lemma 5 (mechanism of evolution [7, 8]). For $u\upsilon w \in F(T)$, $\mathrm{FLEX}_i(T)$ *survives under* $\mathrm{sp}\langle v,u,w\rangle : T \mapsto T'$ *if and only if* $u\upsilon w \in F(\mathrm{FLEX}_i(T))$. *When* $u\upsilon w \notin F(T)$, $\mathrm{FLEX}_i(T)$ *survives if and only if it has no face* υxy *such that the corners* $\langle x,\upsilon,y\rangle$ *and* $\langle u,\upsilon,w\rangle$ *cross each other in* T. ■

Lemma 6 (Lawrencenko [7]). *There are, in all, two triangulations of* $\tilde{\Sigma}_1$, *up to isomorphisms, resulting from the triangulation* IT_1 [Fig. 1(a)] *by a single splitting, namely:*

$$IT_1^a = \mathrm{sp}\langle 0,2,4\rangle(IT_1),\ IT_1^b = \mathrm{sp}\langle 0,2,3\rangle(IT_1). \quad ■$$

Lemma 7 (Lawrencenko [7]). *The graph of each triangulation* IT_1, IT_2, IT_1^a *and* IT_1^b *triangulates the projective plane uniquely up to isomorphisms.*

Proof. Consider first IT_1 and IT_2. Since the property of a triangulation to be irreducible is in fact a property of its graph, any triangulation with the graph of an irreducible triangulation is also irreducible. On the other hand, IT_1 and IT_2 are all irreducible triangulations of $\tilde{\Sigma}_1$ and, moreover, they have nonisomorphic graphs, and the result follows.

Triangulations IT_1^a and IT_1^b are treated similarly to each other. Let us consider IT_1^a. Its edge arisen from vertex 2 of IT_1 under the splitting can be shrunk in *each* triangulation $G(IT_1^a) \to \tilde{\Sigma}_1$, since this edge occurs in exactly two 3-cycles. Furthermore, shrinking this edge always results in triangulation IT_1, because the effect of the restriction of shrinking an edge in a triangulation T to its graph $G(T)$ is independent of the particular choice of T among the triangulations with this graph. Hence, each triangulation $G(IT_1^a) \to \tilde{\Sigma}_1$ is isomorphic to one resulting from IT_1 by a single splitting and applying Lemma 6, along with the observation that the graphs $G(IT_1^a)$ and $G(IT_1^b)$ are nonisomorphic, completes the proof. ■

Lemma 2, in fact, states the equality, $1+\operatorname{flex}(IT_1, \tilde{\Sigma}_1) = 12$, since $G(IT_1) = K_6$. In the proof of the next lemma, we establish the same equality by a general method based on the orbit-stabilizer formula (4).

Lemma 8 (Lawrencenko [7]). $1+\operatorname{flex}(T, \tilde{\Sigma}_1) = 12, 6, 6, 2$, *for* $T = IT_1, IT_2, IT_1^a, IT_1^b$, *respectively*.

Proof. We apply formula (4). By Lemma 7, $N=1$. For illustration, consider $T = IT_1$; the other equalities can be checked similarly. Observe that $\operatorname{Aut}(G(IT_1))$ is the symmetric group S_6. Observe also that $\operatorname{Aut}(IT_1)$ acts transitively on $V(IT_1)$ and the stabilizer of each vertex is the dihedral group D_5, whence $\operatorname{Aut}(IT_1)$ is the alternating group A_5. Applying formula (4) gives

$$1+\operatorname{flex}(IT_1, \tilde{\Sigma}_1) = \frac{|\operatorname{Aut}(G(IT_1))|}{|\operatorname{Aut}(IT_1)|} = \frac{|S_6|}{|A_5|} = 12. \quad ■$$

4. PROOFS

Proof of Theorem 2. A *minor* of an embedding Ψ in Σ is an embedding isomorphic to one obtained from Ψ by repeatedly applying two operations: edge deletion and edge contraction (corresponding to the collapsing of the edge that identifies its endpoints). A polyhedral embedding is *minor-minimal*, if no minor of that embedding is polyhedral. The property of being polyhedral is closed upward under minor relation, and Robertson and Seymour's argument [13] on graph minors, guarantees the finiteness of minor-minimal polyhedral embeddings in Σ, in number, up to isomorphisms.

Clearly, when an edge of some polyhedral graph G polyhedrally embedded in Σ is deleted or contracted, each embedding $G \to \Sigma$ transforms to another embedding of Σ; furthermore, if some two embeddings $G \to \Sigma$ were distinct before deletion or contraction, they are still distinct after the performance of either of these operations. Therefore, if Ψ and Ψ' are two polyhedral embeddings on Σ and Ψ is a minor of Ψ', we have the following inequality:

$$\operatorname{flex}(\Psi', \Sigma) \le \operatorname{flex}(\Psi, \Sigma). \tag{8}$$

This inequality together with the finiteness of minor-minimal embeddings implies a constant upper bound on the flexibilities of the embeddings $G \to \Sigma$, over $G \in \mathrm{PG}(\Sigma)$, and applying Inequality (6) proves the existence of a constant upper bound on the ratio $|\mathrm{Aut}(G)|/|E(G)|$, over $G \in \mathrm{PG}(\Sigma)$. Be Inequality (2), this ratio is also bounded above by a constant when taken over $G \in \mathrm{TG}(\Sigma)$. The theorem follows. ■

Proof of Theorem 3. Let us prove first that $W_T(\tilde{\Sigma}_1) = 48$. By Lemma 6, any triangulation of $\tilde{\Sigma}_1$ is either isomorphic to IT_1 or can be obtained from IT_1^a, IT_1^b, or IT_2 by a sequence (maybe empty) of splittings. The desired equality is proved by a combination of Lemma 8, Inequality (6), and the following inequality:

$$\mathrm{flex}(T', \Sigma) \le \mathrm{flex}(T, \Sigma), \tag{9}$$

for any pair of triangulations T and T' of a fixed surface Σ, where T' is obtained from T by a sequence of splittings. Inequality (9) can be derived from Inequality (8), or from Eq. (7).

The equality $W_T(\Sigma_1) = 240$ can be derived similarly, using the torus analogs of Lemmas 6, 7 and 8, which exist and may be found in [6]. We omit the details in the torus case.

To prove that $W_P(\tilde{\Sigma}_1) = 48$, apply Inequality (6), along with a result of Vitray [15] implying that, if $G \in \mathrm{PG}(\tilde{\Sigma}_1)$, then G has fewer than twelve distinct embeddings in the projective plane unless G is K_6, which has exactly 12 embeddings (Lemma 2). This completes the proof. ■

Lemma 9 (Chen and Lawrencenko [3]; Negami, Nakamoto and Tanuma [10]). *There exists a constant upper bound on the number of flexible faces in a triangulation of a fixed surface.*

Proof. This is a combination of Lemmas 3 and 4. ■

The following is a useful observation, which is obvious.

Lemma 10. *The action of any automorphism of a triangulation T on the face set $F(T)$ sends flexible* [*respectively, rigid*] *faces onto flexible* [*rigid*] *faces of T.*

Proof of Theorem 4. The first factor of the upper bound (5), for $\Psi = T$, is bounded above by a constant, $C_1 = C_1(\Sigma)$, by Lemma 3 along with Inequality (9). The second factor is also bounded above by a constant, $C_2 = C_2(\Sigma)$, by a combination of Lemmas 1, 9 and 10. More precisely, an automorphism of T always fixes the set of the flags of T containing flexible faces (i.e., permutes the flags between themselves), by Lemma 10, and, moreover, is uniquely determined by its effect on any one such flag, by Lemma 1. Therefore, $|\mathrm{Aut}(T)|$ does not exceed the number of flags of T with a flexible face, which number equals, obviously, 6 times the number of flexible faces in T, which number is bounded by Lemma 9. It follows that $|\mathrm{Aut}(G)| \le C_1 C_2$. Therefore, the bound (1) holds for all graphs $G \in \mathrm{TG}(\Sigma)$ with at least $\left\lceil \frac{C_1 C_2}{4} \right\rceil$ edges, or, by Euler's equation, with at least $C(\Sigma) = \left\lceil 1 + \frac{C_1 C_2}{12} \right\rceil$ vertices. ■

To *fix a face* (*deliberately*) in a triangulation T means to make that face rigid by merely deleting the flexes of T that do not contain it. On the other side, fixing a face may turn some other face xyz into a rigid face, which is the case when all the flexes remaining after the deletion contain xyz; we say, then, that face xyz is *fixed automatically*. Triangulating a face *nontrivially* means replacing that face with a triangulation of itself with at least one vertex inside but without new vertices added to its boundary. By Lemma 4 with Note 5, nontrivially triangulating a face makes that face rigid, more precisely, all the faces in that ex-face are rigid, with respect to the resulting triangulation. It is also clear from Lemma 5 that fixing a face of T is equivalent (from the rigidity-flexibility viewpoint) to splitting one of its actual corners followed by repeatedly splitting the corners inside that (ex-)face; this process retriangulates the iterior of that face (without adding more vertices to its boundary) and preserves its rigidity.

Since only the projective plane is under consideration from now on, we suppress "$\tilde{\Sigma}_1$" in the notations in the remainder of this section.

A *bouquet* is defined to be a simplicial 2-complex, which is a subcomplex of the triangulation IT_2 (regarded as a 2-complex) determined by a pair of faces whose intersection is a single vertex of degree 4; thus, there are six bouquets in IT_2.

Lemma 11 (Lawrencenko [8]). *A triangulation T, not IT_1 or IT_2, of the projective plane is 2-flexible if and only if T is isomorphic to a triangulation obtained from the triangulation IT_1* [Fig. 1(a)] *or IT_2* [Fig. 1(e)] *by retriangulating the faces, called $*$-faces, in one of the canonical collections of edge-disjoint faces; these canonical collections are split into three groups, as follows:*

(i) $\{130, 514, 352\} \subset F(IT_1)$, $\{021, 243, 405\} \subset F(IT_1)$, $\{045, 023, 643, 625\} \subset F(IT_2)$, $\{045, 023, 642, 635\} \subset F(IT_2)$;

(ii) $\{130, 514\} \subset F(IT_1)$, $\{045, 023\} \subset F(IT_2)$;

(iii) $\{130\} \subset F(IT_1)$.

Furthermore, the above-listed faces must be retriangulated nontrivially, except possibly one, and only one, of each pair of the faces forming a bouquet in IT_2 (*fixing one face of the bouquet fixes the other face automatically*). *For the collections of $*$-faces in group (i), (ii), or (iii), we have* $\mathrm{flex}(T) = 2$, 3, or 5, *respectively.* ■

Three important simplicial 2-complexes are determined by the faces shaded in Fig. 1, namely: the *bunch of four triangles*, BT, shaded in Fig. 1(a); the *bunch of three squares*, BS, shaded in Fig. 1(c); and the *bunch of three bouquets*, BB, shaded in Fig. 1(e). Observe that the triangulation IT_2 can be obtained from the triangulation IT_1 by retriangulating the underlying space $|BS|$ of BS, more precisely, IT_2 contains the following three “squares”: 2435, 1405, and 1203 [Fig. 1(e)]. By retriangulating the underlying space $|B|$ of a 2-subcomplex B in a triangulation T, we mean the replacement of B by a triangulation B' of $|B|$ so that the boundaries of B and B' are identical and the process does not result in multiple edges.

The structural characterization of flexible triangulations of the projective plane is established in [3]. In the present article, we reproduce the proof of this result for the sake of completeness.

Lemma 12 (Chen and Lawrencenko [3]). *All* 1-*flexible triangulations of the projective plane, up to isomorphisms, can be generated from the triangulations* IT_1 [Fig. 1(a) *and* (c)] *and* IT_2 [Fig. 1(e)] *by retriangulating the underlying space of one of the canonical bunches,* BT, BS, *or* BB, *without adding new vertices to their boundaries and without producing multiple edges.*

Proof. Observe first that the “common parts” of the pairs of distinct triangulations $(\mathrm{a, b})$, $(\mathrm{c, d})$ and $(\mathrm{e, f})$ of Fig. 1 are exactly the bunches BT, BS, and BB, respectively. It follows that any triangulation obtained by retriangulating $|BT|$ or $|BS|$ in IT_1, or $|BB|$ in IT_2, is indeed flexible. Therefore, our job is to prove that *any* flexible triangulation can be obtained in this fashion. Recall that each triangulation of the projective plane can be obtained from IT_1 or IT_2 by a sequence of splittings. To characterize the splittings of the sequence under which the resulting triangulation is still flexible, we examine the evolution of the sets $\{\mathrm{FLEX}_i(IT)\}$ more delicately, for the triangulations $IT \in \{IT_1, IT_2\}$. The purpose of our next steps is to come to the following conclusion: once a splitting of the sequence affects some two neighboring faces of IT, a whole copy of the bunch BS is automatically fixed in IT, but the triangulation is still flexible; furthermore, the next splittings of the sequence either retriangulate the underlying space of the bunch BS fixed or result in a rigid triangulation.

Similarly to fixing a face, fixing two neighboring faces is equivalent to retriangulating the interior of their union, which again can be done by repeatedly splitting appropriate corners. For the sake of simplicity, the reader may imagine that a single vertex is places in every face that is fixed (deliverately or automatically), the “center” of the face, and joined to each vertex in its boundary.

We want to generate all flexible triangulations up to isomorphisms. So, to verify the statements below, it is helpful to bear in mind that the automorphism group of IT_1 is flag-transitive; see [7, 8]. Also, each vertex of degree four [respectively, six] of IT_2 can be sent to vertex 6 [vertex 2] by an appropriate automorphism of IT_2; furthermore, the stabilizer of vertex 6 [vertex 2] acts transitively on the set of edges $\{62, 64, 63, 65\}$ [on the sets $\{20, 26, 21\}$ and $\{23, 24, 25\}$].

Using Lemma 5 and Table 1, the reader may verify that, if we fix any collection of pairwise edge-disjoint faces in IT, we have a triangulation that is still flexible and can be obtained either from IT_1 by fixing appropriate faces in a copy of the bunch BT or in a copy of the possibly retriangulated bunch BS, or from IT_2 by fixing some faces in a copy of the bunch BB. We may need to retriangulate $|BS|$ in the case of $IT = IT_2$; for instance, if we fix faces 054, 032, 634 and 652 in IT_2 [Fig. 1(e)]. Furthermore, if we fix any pair of neighboring faces in IT, we always have a flexible triangulation, which can be obtained from IT_1 by fixing a whole bunch BS, i.e., all the faces of a possibly retriangulated copy of BS.

Assume that υ is one of the original vertices of IT and that u_i $(i = 1, 2)$ is one of the original vertices or the center of one of the original faces of IT. Then the reader may verify that, if the edges υu_1 and υu_2 are in nonneighboring faces, then the triangulation $\mathrm{sp}\langle u_1, \upsilon, u_2\rangle(IT)$ is rigid unless υ is a vertex of degree four in IT_2, say vertex 6, in which event $\mathrm{sp}\langle u_1, \upsilon, u_2\rangle$ fixes the whole bunch BS consisting of retriangulated "squares" 2435, 1405, and 1203 [Fig. 1(e)].

Under fixing the whole bunch BT shaded in IT_1 [Fig. 1(a)], only two flexes survive, namely: $\mathrm{FLEX}_0(IT_1)$ and $\mathrm{FLEX}_1(IT_1)$ [Figs. 1(a) and (b)]. Furthermore, fixing the whole bunch BS shaded in IT_1 [Fig. 1(c)] destroys all the flexes except $\mathrm{FLEX}_0(IT_1)$ and $\mathrm{FLEX}_2(IT_1)$ [Figs. 1(c) and (d)]. Similarly, after fixing the whole bunch BB shaded in IT_2 [Fig. 1(e)], only $\mathrm{FLEX}_0(IT_2)$ and $\mathrm{FLEX}_1(IT_2)$ are left [Figs. 1(e) and (f)]. Furthermore, with B designating any one of the three bunches fixed, it is routine to verify that any splitting $\mathrm{sp}\langle u_1, \upsilon, u_2\rangle$ such that $\langle u_1, \upsilon, u_2\rangle \not\subset |B|$ makes the triangulation rigid. Now the lemma is obvious. ■

Since fixing a single face [respectively, fixing a whole bunch BS] in IT_1 reduces the size of its flex set to 6 [to 2], we are led to the following result (a similar result, for maximum connectivity $\kappa = 5$, is derived by Negami via a different method [11]).

Corollary 1 (Lawrencenko [7]). *Let T be an arbitrary triangulation of the projective plane, not IT_1 or IT_2. Then, if the connectivity $\kappa(G(T)) = 3,\ 4,$ or 5, we have $|\{\mathrm{FLEX}_i(T)\}| \le 6,\ 2,$ or 1, respectively. Furthermore, for each κ, equality holds on infinitely many triangulations.* ■

Lemma 13. *A triangulation T of the projective plane has flexibility* 1 *if and only if T is isomorphic to a triangulation obtained from IT_1 or IT_2 by retriangulating the underlying space of the bunch BT, BS or BB* (*without adding new vertices to their boundaries and without producing multiple edges*)*, until the underlying space contains only rigid faces* (*of the resulting triangulation*).

Proof. *Sufficiency*: Assume, for certainty, that T is obtained from IT_1 by retriangulating the underlying space $|BT|$; the cases of the other two bunches are considered similarly. Then we have a nontrivial flex of T, with the bunch BT unchanged; see Fig. 1(b). Hence, $\mathrm{flex}(T) \ge 1$. On the other hand, $\mathrm{flex}(T) > 1$ would contradict Lemma 11, since none of the collections of $*$-faces contains the bunch BT as a subcomplex. Hence, $\mathrm{flex}(T) = 1$.

Necessity: By Lemma 12, T can be obtained by retriangulating one of the bunches. Furthermore, as above, we can construct a nontrivial flex of T with the corresponding bunch unchanged. Hence, if T did contain a flexible triangle in the retriangulated bunch, we would have a second nontrivial flex, which, however, would contradict the hypothesis $\mathrm{flex}(T) = 1$. ■

Proof of Theorem 5. Let T be a triangulation with graph G. We have three cases to consider.

Case 1: $\text{flex}(T) \geq 2$;

Case 2: $\text{flex}(T) = 1$;

Case 3: $\text{flex}(T) = 0$.

Consider Case 1. By Lemma 11, T is obtained from $IT \in \{IT_1, IT_2\}$ by retriangulation some collection of $*$-faces, denote it by MF, and every face of T in the retriangulated MF is rigid. Furthermore, the reader may verify, as in the Proof of Lemma 12, that fixing the faces of MF never fixes any face not in MF automatically, whence every face not in MF is flexible. We define a $*$-*boundary flag* to be a flag $\{u, u\upsilon, u\upsilon w\}$ of T such that υ is a vertex on the boundary of a $*$-face, which is a member of MF, $u\upsilon$ is an edge on the boundary of the chosen $*$-face, and υuw is the face of T inside the chosen $*$-face. Then we proceed as in the Proof of Theorem 4: Since the boundaries of the members of MF are obviously edge-disjoint, an automorphism of T always fixes the set of $*$-boundary flags of T (i.e., permutes them between themselves), by Lemma 10, and, moreover, is uniquely determined by its effect on any one such flag, by Lemma 1. Therefore, $|\text{Aut}(T)|$ does not exceed the number of marked boundary flags, which number equals, obviously, 6 times the number of $*$-faces. We now apply Inequality (5) as follows:

$$|\text{Aut}(G(T))| \leq (1 + \text{flex}(T)) \cdot |\text{Aut}(T)| \leq 6 \cdot (1 + \text{flex}(T)) \cdot \#(\text{markedfaces}) < 84 .$$

This constant bound of 84 is checked straightforwardly for each collection of $*$-faces in Lemma 11. On the other hand, 84 is 4 times the number of edges in a triangulation of the projective plane with 8 vertices.

In Case 2, we proceed similarly; more precisely, we apply Lemma 13 with the 4 triangles of the bunch BT, or the 3 squares of BS, or the 3 bouquets of BB, considered instead of the $*$-faces (respectively):

$$|\text{Aut}(G(T))| \leq (1 + \text{flex}(T)) \cdot |\text{Aut}(T)|$$

$$\leq \begin{cases} 6 \cdot (1 + \text{flex}(T)) \cdot \#(4 \text{ triangles}) = 6 \cdot 2 \cdot 4 < 84, \\ 8 \cdot (1 + \text{flex}(T)) \cdot \#(3 \text{ squares}) = 8 \cdot 2 \cdot 3 < 84, \\ 12 \cdot (1 + \text{flex}(T)) \cdot \#(3 \text{ bouquets}) = 12 \cdot 2 \cdot 3 < 84. \end{cases}$$

In Case 3, the proof of Inequality (1) given for the spherical graphs in Section 2 readily applies. ■

5. WEINBERG LIMIT SUPERIORS

For a fixed surface Σ, the definition of *Weinberg limit superiors* $\overline{W}_P(\Sigma)$ and $\overline{W}_T(\Sigma)$ is obtained from Definition 1 of Weinberg bounds $W_P(\Sigma)$ and $W_T(\Sigma)$, respectively, by replacing “sup” with “lim sup”. The finiteness of $\overline{W}_P(\Sigma)$ and $\overline{W}_T(\Sigma)$ follows from the finiteness of $W_P(\Sigma)$ and $W_T(\Sigma)$, respectively.

Conjecture 3. $\overline{W}_P(\Sigma_0)=\overline{W}_T(\Sigma_0)=\frac{4}{3}$; $\overline{W}_P(\tilde{\Sigma}_1)=\overline{W}_T(\tilde{\Sigma}_1)=\frac{2}{3}$.

As a step towards proving this conjecture, in the remainder of this section we establish the following inequalities:

$$\overline{W}_T(\Sigma_0)\geq\frac{4}{3},\quad \overline{W}_T(\tilde{\Sigma}_1)\geq\frac{2}{3}. \tag{10}$$

To establish the first of these inequalities, take two congruent pyramids with n-gonal bases, $n\geq 3$, and identify their bases (vertices being identified with vertices, edges with edges). Clearly, the resulting graph has $3n$ edges, and its automorphism group has (for n different from 4) order $4n$.

To establish the second of Inequalities (10), let n be an odd integer, $n\geq 7$. Let B_n be a 2-disk in Euclidean plane with $2n$ vertices evenly spread on its boundary circle. Label the vertices by 1 through n and once more by 1 through n as they occur on the circle, say clockwise, in such a way that the antipodal vertices obtain the same label. Think of B_n as a result of cutting the projective plane $\tilde{\Sigma}_1$ around a nontrivial cycle, of length n, denote it by C_n, which cycle is laid on the boundary of B_n. Now, starting from vertex 1, proceed around the boundary of B_n clockwise and join vertex 1 to vertex 3, vertex 3 to vertex 5, vertex 5 to vertex 7, and so forth, finally, vertex $n-1$ to the initial vertex 1. Since n is odd, we thus obtain another cycle, C'_n, spanning the vertices 1 to n. Finally, place one more vertex in the center of B_n, join it to each vertex of C'_n, and thereby obtain a triangulation, PP_n of $\tilde{\Sigma}_1$ with the graph having connectivity 5. By Corollary 1, every 5-connected graph in $\mathrm{TG}(\tilde{\Sigma}_1)$, except K_6, admits a combinatorially unique triangulation of the projective plane, hence $\mathrm{Aut}(G(PP_n))=\mathrm{Aut}(PP_n)$, and, hence, we have

$$\frac{|\mathrm{Aut}(G(PP_n))|}{|E(G(PP_n))|}=\frac{|\mathrm{Aut}(PP_n)|}{|E(PP_n)|}=\frac{|D_n|}{3n}=\frac{2n}{3n}=\frac{2}{3}.$$

ACKNOWLEDGMENTS

The authors are pleased to acknowledge the useful discussions with Professor Dan Archdeacon and are grateful to the anonymous referee for helpful comments in improving the presentation.

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