Polyhedral suspensions of arbitrary genus

Serge Lawrencenko

Department of Mathematics
Moscow Institute of Steel and Alloys
4 Leninsky Prospect, Moscow 119049
Russia

lawrencenko@hotmail.com

Abstract. A new class of polyhedra is discovered—bipyramids of arbitrarily prescribed genus. A two-dimensional generalization of Fáry's Theorem is established. A purely combinatorial definition of a polyhedral suspension is given. A new regular 2-dimensional polyhedron is constructed in four dimensions.

Short running title: Polyhedral suspensions.

2000 AMS Subject Classification: Primary: 57M20; *Secondary:* 05C10, 90C57, 52B05, 52B70, 57M15, 51M20, 51M04, 55P40.

Key words and phrases: regular polyhedra, simplicial complexes, planarity, graph embedding, maximum genus.

1. Introduction

The simplest example of a (spherical) polyhedral suspension is provided by the triangular bipyramid, on the left of Figure 1. A toroidal polyhedral suspension, on the right of Figure 1, was discovered by the author [11, 12] in 1983.

Some of the results of [11, 12] can be also found in [1]. By the way, in [11, 12, 1], a toroidal polyhedron with 7 vertices in 3-space was rediscovered, also known as Császár's torus [3].

The suspensions in Figure 1 are slightly inclined with their highest vertices ("north poles") toward the reader to make a little rift visible in the toroidal polyhedron on the right. While the suspension on the left looks very unassuming, the one on the right is quite exotic. This paper is the author's attempt to understand the structure of such exotic polyhedra.

In Section 5 we establish the existence of polyhedral suspensions of arbitrarily prescribed genus. In Section 4 we discover a new regular 2-dimensional polyhedron in 4-space (Euclidean). In Section 2 we establish a two-dimensional generalization of Fáry's Theorem. In Section 3 we decide in combinatorial terms whether a given abstract 2-complex (simplicial) can be realized in 3-space as a geometric polyhedral suspension of given genus. In Section 6 we derive some graph-theoretical corollaries of our construction, in particular: there exist planar graphs that can be 2-cell embedded on surfaces of arbitrarily large genus with all but two regions triangular.

The book by Gross and Tucker [7] is a good source for topological graph theory and 2-complexes. By a 2-complex we mean a 2-dimensional finite simplicial complex. By a polyhedron we mean one embedded in Euclidean space without self-intersections and without unbounded faces. With any 2-dimensional polyhedron P with triangular faces, one associates the geometric 2-complex of P; it consists of the vertices, edges, and triangles of P. Let $G(P)$ denote the *graph* (also called the *1-skeleton*) of P; it consists of the vertices and edges of P. Let $G(K)$ denote the 1-skeleton of a 2-complex K; we usually call it the *graph of* K. The *star* of a vertex v in K, denoted $\mathrm{star}(v)$, is the minimal subcomplex of K that contains each simplex of K that is incident with v. The *link* of a vertex v, denoted $\mathrm{link}(v)$, in K is the maximal subcomplex of $\mathrm{star}(v)$ that does not contain v itself. By a surface we mean a 2-dimensional closed (that is, compact and without boundary) surface. Specifically, by Σ_g we denote the connected orientable surface of genus g.

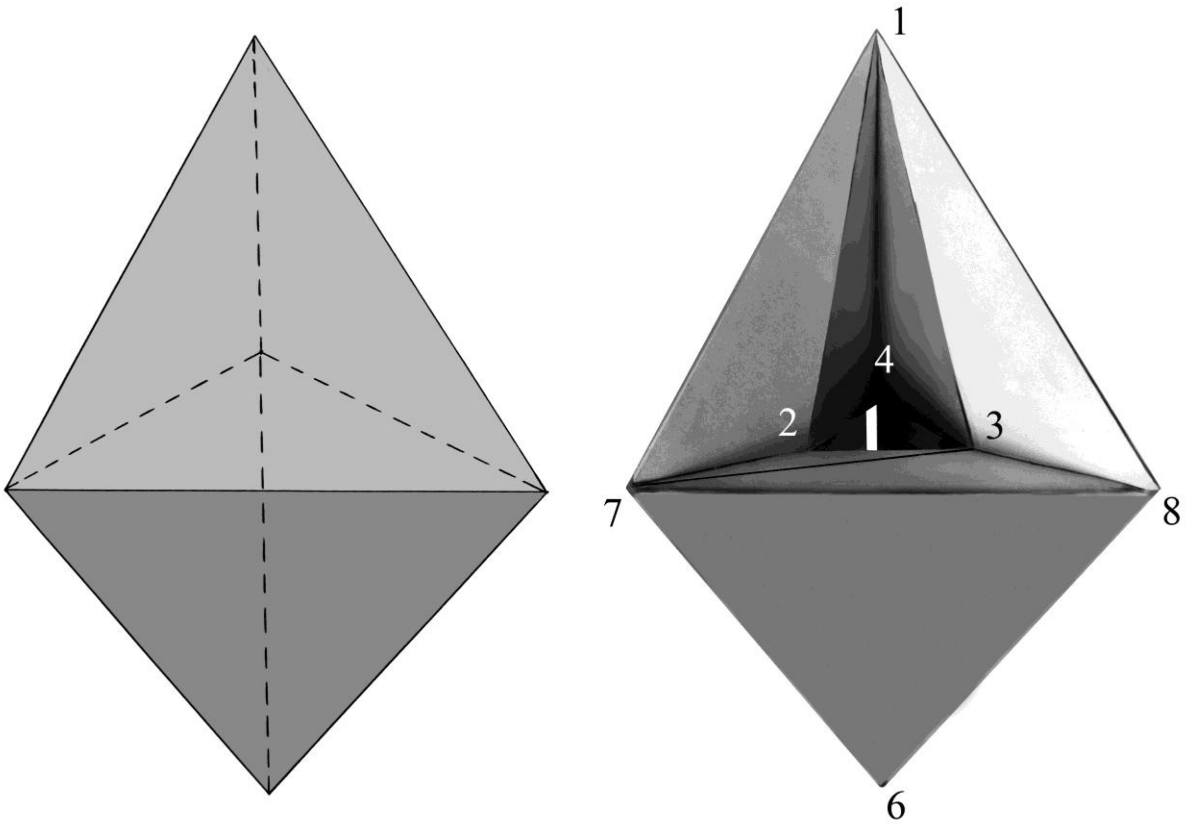

Figure 1. Spherical and toroidal suspensions.

Definition 1. By a *geometric polyhedral suspension* we mean a 2-dimensional polyhedron C with bounded triangular faces in Euclidean 3-space, that is homeomorphic to Σ_g and satisfies the following two conditions:

(i) All but two of the vertices of C lie in one plane called the "equatorial plane".
(ii) The two vertices not in the equatorial plane are nonadjacent in the graph of C.

The simplicial complex determined by the equatorial vertices, edges, and triangles of C is called the "equator" of C and is denoted by $\mathrm{eqtr}(C)$. The two vertices not in the equatorial plane are located in different half spaces determined by the equatorial plane and are called the "north pole", N, and "south pole" S. Thinking of the equatorial plane as horizontal, we'll say that N and S are placed above and below that plane respectively. ■

It is not true that the equator is necessarily 1-dimensional in the spherical case. For a counterexample, one can distort the icosahedron in 3-space so that it becomes a spherical polyhedral suspension with a 2-dimensional washer-shaped equator.

Therefore the structure of a polyhedral suspension C gives rise to a stratification of the underlying surface Σ_g into 3 strata corresponding to the 3 simplicial complexes: the equator, the northern cone, and the southern cone:

$$C = \mathrm{eqtr}(C) \cup \mathrm{star}(N) \cup \mathrm{star}(S) .$$

This stratification will turn out to be important.

2. Planarity of 2-complexes

In this section we address the general issue of planarity of 2-complexes. This is closely related to the structure of polyhedral suspensions as we'll see in the next section.

A 2-complex K is said to be *topologically planar* if it can be embedded in the plane in the topological sense—that is, in such a way that the 1-simplexes of K are represented by interiorly disjoint Jordan curves and the 2-simplexes are represented by bounded triangular regions of the graph embedding. Furthermore, K is *geometrically planar* if it can be realized as a geometric 2-complex in the plane—that is, with all of its 1-simplexes represented by straight line segments and with all of its 2-simplexes represented by bounded triangles.

Classically, topologically planar 1-complexes are characterized by Kuratowski [10] in graph-theoretical terms. Later Fáry [4] proves that each topologically planar 1-complex is geometrically planar. We now establish a two-dimensional generalization of Fáry's Theorem.

Theorem 1. *A 2-complex K is geometrically planar if and only if K is topologically planar.*

Proof: We only have to prove that the topological planarity of K implies its geometric planarity. We'll use Steinitz's Theorem [20] implying that any topological triangulation T of the 2-sphere can be geometrically realized as the boundary 2-complex of a convex 3-polytope in 3-space whose 2-faces represent the triangles of T so that the angle between any pair of adjacent 2-faces is strictly between 0 and π.

Let K be topologically embedded in the plane. We may assume that K has at least 4 vertices. Subdivide nontriangular regions (if any) by successively drawing additional interiorly disjoint Jordan curves until a triangulation, T, is attained. Let Q be a strictly convex 3-polytope whose boundary 2-complex ∂Q is isomorphic to T and let f be the 2-face of the polyhedron ∂Q corresponding to the unbounded triangular region of T. Now place ∂Q in between some two parallel planes p_1 and p_2 so that f lies in one of those planes, in p_2 say. Pick a point, O, in the other half space determined by p_2, sufficiently close to f. The phrase "sufficiently close" means that if L is the cone determined by O and the boundary 1-complex ∂f of f, then $L \cap Q = \partial f$. Then the stereographic projection with the center at O onto the plane p_1 maps ∂Q (with the interior of f removed) into a desired geometric embedding of K in p_1, only we need to remove the 1-simplexes corresponding to the extra edges that we drew in the beginning of the proof. ■

By Theorem 1, there is no gap between topological planarity and geometric planarity for 2-complexes, so hereafter we shall speak simply of "planarity". (In fact, a stronger result can be derived from the above proof: Given a topological embedding of a 2-complex K in the plane, there is a geometric embedding of K in the plane isotopic to the given topological embedding.)

As we'll see in the next section, it is important to have a combinatorial characterization of planar 2-complexes. A possible approach is to reduce the planarity question for a 2-complex to testing the planarity of its graph, since we can test the latter in combinatorial terms, using Kuratowski's Theorem. It is not generally true that a 2-complex is planar whenever its graph is planar. For a counterexample, removing all the six 2-simplexes from the triangular bipyramid on the left of Figure 1 and filling its equatorial cycle with a new 2-simplex lead to a nonplanar 2-complex having a planar graph of connectivity 3 and only one 2-simplex. However, a simplicial 2-complex not homeomorphic to the 2-sphere is planar whenever its graph is planar and 4-connected.

We attribute to Gross and Rosen the theorem stating that planarity of a simplicial 2-complex can be tested in purely combinatorial terms. For, it is their planarity criterion [6] that a 2-complex K is planar if and only if the following three conditions are satisfied: (1) the star of every vertex of K is planar, (2) the graph of the first barycentric subdivision of K is planar, and (3) K contains no subcomplex homeomorphic to the 2-sphere. Furthermore, conditions (1) and (2) are necessary and collectively sufficient for K to be embeddable in the 2-sphere; see [6, 14]. It is also a result of Gross and Rosen [5] that the problem of reducing the planarity question for a given 2-complex to testing the planarity of a graph is solvable in linear time in the number of vertices of the 2-complex.

Interestingly, Mohar [14, 15] has also established a toroidality criterion: A 2-complex K embeddable on Σ_g for some g is toroidal if and only if the graph of the third barycentric subdivision of K is a genus-one graph.

Open Problem 1. Mohar [15] raised the question about whether the first, or second, barycentric subdivision suffices.

3. Definitions of an abstract polyhedral suspension

What is the minimum number of interiorly disjoint closed disks such that the removal of their interiors from Σ_g ($g \neq 0$) leaves a remainder stretchable onto a plane without breaking or overlapping? The answer is 2, which follows from the existence of polyhedral suspensions having arbitrarily prescribed genus (to be constructed in Section 5). Inspired by this topological metaphor, we propose a new combinatorial invariant as follows.

The *spatiality* of a simplicial 2-complex K, denoted $s(K)$, is the least number of pairwise nonadjacent vertices, forming a *planarizing set* for K, whose removal (together with the incident simplexes) from K leaves a simplicial complex planar. Notice that if the carrier of K is homeomorphic to the 2-sphere, then $s(K)=1$, where every one-vertex set is planarizing. But if it is homeomorphic to Σ_g ($g \neq 0$), then $s(K) \geq 2$.

Since every triangulation of the 2-sphere with at least 5 vertices has a pair of nonadjacent vertices, it can be realized as a geometric polyhedral suspension in accordance with Definition 1. Therefore any such triangulation can be regarded as an *abstract polyhedral spherical suspension*. In what follows we shall focus on the nonspherical case.

Definition 2. An *abstract polyhedral nonspherical suspension* is an abstract simlicial 2-complex C with spatiality $s(C)=2$, whose carrier is homeomorphic to a closed orientable surface. Any 2-vertex planarizing set for C can be regarded as the set of two "poles" N and S, and the maximal simplicial subcomplex of C containing neither N nor S serves as "equator". ■

Definition 2 is combinatorial topological. For, it exploits the topological notion of homeomorphism and, speaking of the carrier of an abstract 2-complex K, we implicitly assume some geometric realization of it. (Nonetheless, any abstract 2-complex determines the topology of any geometric realization of it; see [7].) Definition 2 will be rephrased in purely combinatorial terms at the end of this section.

Clearly, the abstract 2-complex of any geometric polyhedral suspension (Definition 1) is an abstract polyhedral suspension. The converse is also true.

Lemma 1. *Any abstract polyhedral suspension C can be realized in 3-space as a geometric polyhedral suspension.*

Proof: Let $\{u,v\}$ be a planarizing set for C and let D be the maximal simplicial complex containing none of u and v. Then D is planar and, by Theorem 1, is geometrically realizable in the plane. Place u and v above and below that plane respectively. Now, restoring the simplexes of $\mathrm{star}(u) \cup \mathrm{star}(v)$ makes C a geometric polyhedral suspension. ■

There is a correlation between the spatiality of a 2-complex and its orientability type.

Lemma 2. $s(K) \geq 3$ *for any simplicial 2-complex K whose carrier is homeomorphic to a closed nonorientable surface.*

Proof: By contradiction. Assume to the contrary that K has a two-vertex planarizing set, $\{u,v\}$, whose removal from K leaves a planar 2-complex, L. Place u and v above and below the plane in which L is embedded. Since u and v are nonadjacent, and since the boundary of the star of any vertex in a triangulation of a closed surface is homeomorphic to a circle, restoring the simplexes of K removed would lead to an embedding of K in 3-space, which is impossible. ■

We can now rephrase Definition 2 in purely combinatorial terms. Lemma 2 ensures that deleting the word "orientable" from Definition 2 leaves a definition equivalent to Definition 2. Furthermore, the carrier of K is homeomorphic to a *closed surface* if and only if conditions (iii) and (iv) below hold. For, condition (iii) guarantees the absence of singular vertices and condition (iv) guarantees that the surface is closed. To determine the topological type of that surface, let α_l denote the number of l-simplexes in K, for each l with $0 \leq l \leq 2$. Then the Euler characteristic of K is the integer $\chi(K) = \alpha_0 - \alpha_1 + \alpha_2$, and the Euler-Poincaré formula reads: $\chi(K) = 2 - 2g$, where g is the genus of the surface.

Definition 3. Combinatorial definition of an abstract polyhedral nonspherical suspension. An *abstract polyhedral nonspherical suspension of given genus* g ($g \neq 0$) is an abstract simplicial 2-complex C having spatiality $s(C) = 2$ and Euler characteristic $\chi(C) = 2 - 2g$, and satisfying the following two conditions:

(iii) The link of each vertex v in C forms a Hamilton cycle through the neighbors of v.
(iv) Every 1-simplex of C is incident with precisely two 2-simplexes of C. ■

Definition 3 is equivalent to Definition 2 and can serve as a finite algorithm for testing whether or not a given abstract 2-complex is a polyhedral suspension of given genus g ($g \neq 0$). By Gross', Rosen's, and Mohar's Theorem, $s(K)$ is indeed a combinatorial invariant.

4. Models of the toroidal hexadecahedron

A *bipyramid* is a polyhedral suspension in which the poles are each adjacent to *each* vertex in the equator. It is not true that every polyhedral spherical suspension is necessarily a bipyramid. For a counterexample, the boundary 2-complex of the octahedron *is* a bipyramid, but the distorted icosahedron mentioned in the Introduction is not. It can be proven that a spherical bipyramid has a 1-dimensional equator which is a simple cycle.

We now reproduce the author's construction [11, 12, 1] for building the toroidal bipyramid having 8 vertices, 24 edges, and 16 triangular faces. It is shown on the right of Figure 1. We'll denote the bipyramidal toroidal hexadecahedron of Figure 1 by BTH. In fact, BTH is a geometric 3D realization of the 8-vertex topological triangulation of the 2-torus shown in Figure 2 (opposite sides of the rectangle are identified with equal orientations). It will be called the *abstract toroidal hexadecahedron* and will be denoted by ATH. The pair of vertices 1 and 6 forms a planarizing set for ATH. For, the removal of those vertices together with their incident simplexes from ATH results in a planar 2-complex which can be geometrically embedded in the equatorial plane as shown in Figure 3, where the 2-simplexes are shaded. (In Figure 1 vertex 5 is invisible, but two of the equatorial triangles—$\{7,2,3\}$ and $\{7,3,8\}$—are visible.)

The north pole N (vertex 1) and the south pole S (vertex 6) are placed above and below the equatorial plane, respectively. Adjoin the triangles determined by N and the edges of the "thick basic cycle" $2-4-3-8-5-7-2$ shown by thick lines in Figure 3 as well as the triangles determined by S and the edges of the "thick dashed basic cycle" $2-3-4-8-7-5-2$ shown by thick dashed lines. These cycles are in fact the links of the poles N and S in ATH (and in BTH). Both are Hamilton cycles of the graph $G(\mathrm{eqtr}(\mathrm{BTH}))$. By construction, the so-built polyhedron is indeed a toroidal suspension. That can be also verified by using Lemma 1 along with Definition 3.

We observe from Figure 3 that some two edges of $\mathrm{eqtr}(\mathrm{BTH})$—$\{7,5\}$ and $\{3,4\}$—occur in both Hamilton cycles, since each of those edges is thick and is thick dashed as well. We'll call such edges "dichromatic".

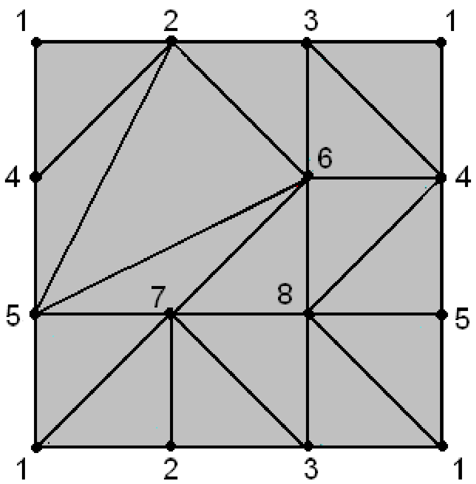

Figure 2. ATH.

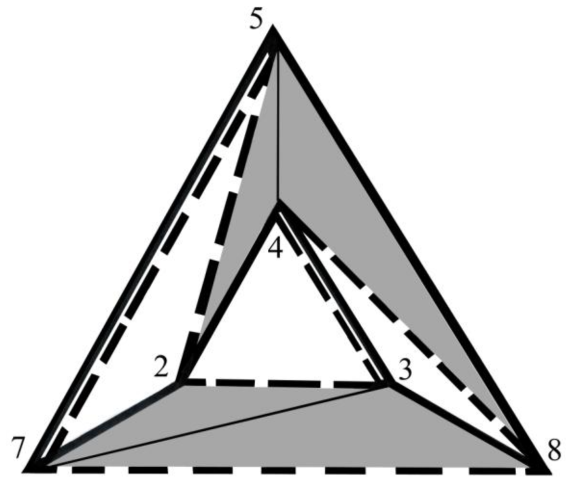

Figure 3. eqrt(BTH).

Notice that ATH is a *regular* triangulation—that is, the degree of each vertex is the same (equal to 6 in this case). Also notice that its graph $G(\mathrm{ATH}) = G(\mathrm{BTH}) = K_{2,2,2,2}$ is the complete fourpartite graph.

Realization of 2-dimensional topological objects proves to be fruitful in Euclidean 4-space rather than in 3-space. For a given polyhedron P in Euclidean n-space, the *(full) symmetry group,* denoted $\mathrm{Sym}(P)$, is the set of all Euclidean isometries of n-space that leave P invariant.

Theorem 2. *There exists a 2-dimensional regular toroidal hexadecahedron,* RTH, *which is a geometric realization of* ATH *in Euclidean 4-space, with regularity properties as follows.*

(v) *All faces of* RTH *are geometrically isometric equilateral triangles.*

(vi) RTH *has no "hidden symmetries" in the sense that* $\mathrm{Aut(ATH)} \cong \mathrm{Sym(RTH)}$*, which means that the automorphism group* Aut(ATH) *is faithfully represented by the symmetry group* Sym(RTH) *in 4-space.*

(vii) Sym(RTH) *is vertex-transitive.*

Proof: The *cross polytope* β_4 is the regular polytope in four dimensions corresponding to the convex hull of the points formed by permuting the coordinates $(\pm 1,0,0,0)$; see [2]. The boundary $\partial\beta_4$ carries the structure of a geometric simplicial 3-complex. In fact, $\partial\beta_4$ is a regular 3-dimensional polyhedron in 4-space. It has 16 threedimensional sides isometric to a regular 3-dimensional tetrahedron. We have $G(\mathrm{ATH}) = K_{2,2,2,2} = G(\partial\beta_4)$. Furthermore, ATH is a subcomplex of the 2-skeleton of $\partial\beta_4$ because $\partial\beta_4$ contains all possible triangles (32 in number). Therefore ATH can be simplicially embedded into $\partial\beta_4$ and thereby realized in 4-space with its 2-simplexes represented by isometric equilateral triangles. Property (v) has been proved. One such realization is obtained by letting the vertices of ATH have the coordinates as follows:

1 $(0,0,0,1)$, 2 $(1,0,0,0)$, 3 $(0,0,-1,0)$, 4 $(0,-1,0,0)$,
5 $(0,0,1,0)$, 6 $(0,0,0-1)$, 7 $(0,1,0,0)$, 8 $(-1,0,0,0)$.

The automorphism group Aut(ATH) is determined in [13]. It can be generated by the involutions $\theta_1 = (35)(47)$ and $\theta_2 = (16)(37)(45)$ together with the cyclic automorphism $\sigma = (15276384)$ (one which permutes all the vertices in a single cycle). Therefore $\mathrm{Aut(ATH)} = (\mathbf{Z}_2 \times \mathbf{Z}_2)\mathbf{Z}_8$ and $|\mathrm{Aut(ATH)}| = 2 \times 2 \times 8 = 32$. Then it can be checked that the generating permutations θ_1, θ_2, and σ are represented (respectively) by the following orthogonal matrices with determinant $+1$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

These matrices correspond to some symmetries of RTH in 4-space, assuming that the vertices of RTH have the coordinates listed in the preceding paragraph. Therefore the symmetry group

$$\mathrm{Sym(RTH)} \subset \mathrm{O}(4) \subset \mathrm{GL}(4, \mathbf{R})$$

provides a faithful representation of Aut(ATH) of degree 4. Property (vi) has been proved.

Property (vii) follows from (vi) and the fact that Aut(ATH) is vertex-transitive (since Aut(ATH) contains a cyclic automorphism). ∎

Note 1. In fact, conditions (vi) and (vii) collectively ensure that in a regular polyhedron the stars of all its vertices are congruent geometric figures.

Note 2. BTH and RTH are two geometric realizations of one abstract 2-complex, ATH, so that BTH is a 3D model but RTH is a 4D model. Interestingly, RTH still carries the structure of a suspension, but now in 4-space. For, the poles—vertices 1 $(0,0,0,1)$ and 6 $(0,0,0-1)$ — lie on a fourth coordinate axis perpendicular to the x, y, and z axes, labeled w, but all the other six vertices are in the equatorial hyperplane $w=0$. However, the 3D equator of RTH is more symmetric than the 2D equator of BTH.

It is a result of Ringel and Youngs [19] that for $n \equiv 0,\ 3$, 4, or 7 modulo 12 the complete graph K_n triangulates Σ_g with $g=(n-3)(n-4)/12$. The following may become a useful starting point for a new research direction.

Lemma 3. *Each topological triangulation T of $\Sigma_{(n-3)(n-4)/12}$ with the complete graph K_n ($n \equiv$ 0, 3, 4, or 7 modulo 12) can be geometrically realized as a 2-dimensional polyhedron $P=P(T)$ in Euclidean n-space with all faces represented by geometrically isometric equilateral triangles.*

Proof: Embed K_n into the 1-skeleton of the cross polytope β_n. Then add the faces of T and get a polyhedron as desired. ■

Open Problem 2. What about hidden symmetries of the polyhedra P as in the statement of Lemma 3? Are there any automorphisms of T that are not realized as symmetries of P?

5. Construction of bipyramids of arbitrarily prescribed genus

Although there may be many bipyramids of given genus, the notation gBTH will henceforth denote specifically the series of bipyramids of genera g to be constructed in this section, assuming a concrete combinatorial structure.

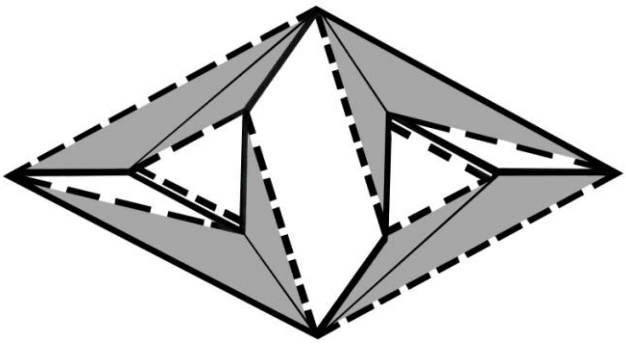

Figure 4. eqtr(2BTH).

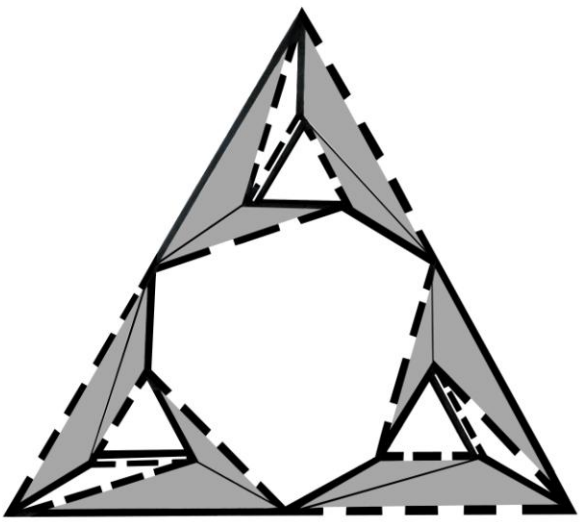

Figure 5. eqtr(3BTH).

Theorem 3. *For each positive integer g, there exists a bipyramid gBTH of genus g.*

Proof: In fact, our job is to construct a bipyramid gBTH of genus g for each given $g \geq 2$. By the "basic template" we'll mean the 2-complex of Figure 3 with the longer dichromatic edge, $\{7,5\}$, removed. To build eqtr(gBTH), take g copies of the basic template and amalgamate them cyclically as indicated in Figure 4 for the double torus 2BTH, then in Figure 5 for the triple torus 3BTH, and so forth.

The rest of the procedure is similar to that in the torus case. To build the bipyramid $g\mathrm{BTH}$, place the north pole N and the south pole S above and below the equatorial plane and proceed to adjoin nonequatorial triangles. For this, extend the thick basic cycle and the thick dashed basic cycle onto the whole equator as shown in Figure 4 for $\mathrm{eqtr}(2\mathrm{BTH})$, then in Figure 5 for $\mathrm{eqtr}(3\mathrm{BTH})$, and so forth. Observe that there are two dichromatic edges in $\mathrm{eqtr}(2\mathrm{BTH})$, three dichromatic edges in $\mathrm{eqtr}(3\mathrm{BTH})$, and, generally, g dichromatic edges in $\mathrm{eqtr}(g\mathrm{BTH})$. As nonequatorial triangles, adjoin the triangles determined by N and the thick edges as well as the triangles determined by S and the thick dashed edges in $\mathrm{eqtr}(g\mathrm{BTH})$.

One can use Definition 3 to check that $g\mathrm{BTH}$ is indeed a suspension of genus g, but now we'd rather apply a topological approach instead. In fact, our construction can be translated into the topological language as follows.

We firstly took g copies of BTH (or, more accurately, g copies of the carrier of the 2-complex ATH). From each of the copies, we cut out the longer equatorial dichromatic edge together with the two triangles that are incident with that edge. We finally connected those g copies, cyclically amalgamating them along the pairs of nonequatorial edges on the boundaries of the holes (one edge of each such pair being incident with N, while the other incident with S) as indicated in Figures 4 and 5. We thus indeed obtained a triangulation $g\mathrm{BTH}$ of Σ_g as the connected sum of g tori BTH (or, more accurately, g copies of ATH). ■

Open Problem 3. Negami [16] proposed the problem of classifying all bipyramids of fixed genus.

6. Properties of bipyramidal graphs

Here we address some properties of the graphs $G(g\mathrm{BTH})$ of the bypyramids $g\mathrm{BTH}$ constructed in the preceding section.

Firstly, the genus $\gamma(G(g\mathrm{BTH}))$ is equal to g because $g\mathrm{BTH}$ is a triangulation of Σ_g.

Secondly, the thickness $\theta(G(g\mathrm{BTH}))$ is equal to 2 for any $g \geq 1$ because

$$G(g\mathrm{BTH}) = G(\mathrm{eqtr}(g\mathrm{BTH})) \cup K_{2,n-2}$$

is the union of two planar graphs. Here $n = |V(G(g\mathrm{BTH}))|$ is the number of vertices of $g\mathrm{BTH}$, and the complete bipartite graph $K_{2,n-2}$ is determined by the 2-partition of the vertex set as follows:

$$V(G(g\mathrm{BTH})) = \{N, S\} \cup V(G(\mathrm{eqtr}(g\mathrm{BTH}))).$$

We come to the following result. (Surprisingly, we have failed to find an account of it in literature.)

Corollary 1. *There exist thickness-two graphs of arbitrarily large genus.* ■

It is also interesting to observe the properties of the equatorial graph $G(\mathrm{eqtr}(g\mathrm{BTH}))$. This graph is planar for any g and admits a 2-cell embedding in Σ_g. One such embedding is attained by removing the poles N and S together with their incident edges from $g\mathrm{BTH}$, which produces two cone-shaped nontriangular regions. It follows that

$$0=\gamma(G(\mathrm{eqtr}(g\mathrm{BTH})))\leq g\leq\gamma_M(G(\mathrm{eqtr}(g\mathrm{BTH}))),$$

where γ_M stands for maximum genus. We therefore rediscover Ringeisen's Theorem [18] which states the existence of planar graphs of arbitrarily large maximum genus. That theorem is in fact stronger as the reader can observe from the title of [18], but we now strengthen it namely in its planar part. For this, we propose a related invariant, the *closed 2-cell maximum genus* of a graph G—that is, the largest h such that G admits a *closed 2-cell embedding* in Σ_h—that is, a 2-cell embedding in Σ_h in which every region is bounded by a simple cycle. We have thereby proved that *there exist planar graphs of arbitrarily large closed 2-cell maximum genus.* In fact, we have proved even a stronger statement:

Corollary 2. *There exist planar graphs that admit a closed 2-cell embedding on surfaces of arbitrarily large genus with all but two regions triangular.* ■

Thirdly, the *crossing number* of a graph G, denoted $\mathrm{cr}(G)$, is the minimum number of crossings in a drawing of G in the plane. As a recent major trend in crossing numbers research, the question of linearity of crossing number has received increasing attention. A family of graphs is said to have *linear* crossing number if there is a constant c such that $\mathrm{cr}(G)\leq cn$ for any graph G in the family, where $n=|V(G)|$. Pach and Tóth [17] prove that toroidal graphs with bounded degree have linear crossing number. Furthermore, Hliněný and Salazar [9] show that a similar approach gives rise to a polynomial-time algorithm for estimating the crossing number of toroidal graphs with bounded degree.

To estimate $\mathrm{cr}(G(g\mathrm{BTH}))$, draw the whole graph $G(g\mathrm{BTH})$ in the equatorial plane as follows. Set N at the center of $\mathrm{eqtr}(g\mathrm{BTH})$ (use Figure 4 or 5) and set S outside $\mathrm{eqtr}(g\mathrm{BTH})$. We need to accommodate the nonequatorial edges of $g\mathrm{BTH}$ in the equatorial plane. Those edges split into two classes: the "northern edges" are incident with N, while the "southern edges" are incident with S. It is not hard to draw the northern edges with 4 crossings with the equatorial edges (on each of the g basic templates) and the southern edges with 4 more crossings with the equatorial edges, which inevitably causes still 2 more crossings between the northern edges with the southern ones. Then $n=|V(G(g\mathrm{BTH}))|=5g+2$ (for $g\geq 2$) and $\mathrm{cr}(G(g\mathrm{BTH}))\leq(4+4+2)g=10g$ force $\mathrm{cr}(G(g\mathrm{BTH}))\leq 10g=2n-4\leq 2n$, where $g\geq 2$.

Corollary 3. *The family of bipyramidal graphs* $\{G(g\mathrm{BTH})\}_{g=2}^{\infty}$ *has linear crossing number with the constant* c *at most 2.* ■

Finally, it is easy to see that, for any $g\geq 2$, the chromatic number $\mathrm{chr}(G(\mathrm{eqtr}(g\mathrm{BTH})))=4$, whence $\mathrm{chr}(G(g\mathrm{BTH}))=5$. It is a related result of Harary, Korzhik, and the author [8] that, for a closed surface of arbitrarily given genus and orientability type, the whole spectrum of the chromatic numbers admissible by that surface is realizable in the class of triangulations of that surface.

Acknowledgments. The author is very grateful to his friends, Professors Seiya Negami and Atsuhiro Nakamoto, for inviting him to deliver a lecture on the subject as part of the 20th (International) Workshop on Topological Graph Theory in Yokohama, held November 24–28, 2008 at Yokohama National University, and to the other participants of the Workshop and also to Professor Idjad Kh. Sabitov for enthusiastic discussions.

References

[1] Bushmelev, A.V., Lawrencenko, S.: Polyhedral tori [in Russian]. Kvant 2, 3-5 (1985), http://kvant.mirror1.mccme.ru/1985/02/mnogogranniki--tory.htm

[2] Coxeter, H.S.M.: Introduction to Geometry (2nd ed.) New York: Wiley 1969

[3] Császár, A.: A polyhedron without diagonals. Acta Sci. Math. (Szeged) 13, 140-142 (1949–1950)

[4] Fáry, I.: On straight line representations of planar graphs. Acta Sci. Math. (Szeged) 11, 229-233 (1948)

[5] Gross, J.L., Rosen, R.H.: A linear-time planarity algorithm for 2-complexes. J. Assoc. Comput. Mach. 20, 611-617 (1979)

[6] Gross, J.L., Rosen R.H.: A combinatorial characterization of planar 2-complexes. Colloq. Math. 44, 241-247 (1981)

[7] Gross, J.L., Tucker, T.W.: Topological graph theory. Mineola New York: Dover Publications, 2001

[8] Harary, F., Lawrencenko, S., Korzhik V.: Realizing the chromatic numbers of triangulations of surfaces. Discrete Math. 122, 197-204 (1993)

[9] Hliněný, P., Salazar, G.: Approximating the crossing number of toroidal graphs. In: T. Tokuyama (ed.) et al.: Algorithms and computation, Proceedings of ISAAC 2007 (Lect. Notes in Computer Science, vol. 4835, pp. 148-159) Berlin: Springer 2007

[10] Kuratowski, K.: Sur le problème des courbes gauches en topologie. Fund. Math. 15, 271-283 (1930)

[11] Lawrencenko, S.: All the twenty-one irreducible triangulations of the torus are realizable as polyhedra in E^3 [in Russian] (Second-place-winning Manuscript, Student Research Paper Contest, Faculty of Mech. & Math.) Moscow: Moscow State University 1983

[12] Lawrencenko, S.: All irreducible triangulations of the torus are realizable as polyhedra in E^3 [in Russian] (Master's Thesis) Moscow: Moscow State University 1984

[13] Lawrencenko, S.: Explicit enumeration of all automorphisms of the toroidal irreducible triangulations and all toroidal embeddings of the labeled graphs of those triangulations [in Russian] (No. 2779 Uk87. 1 Oct 1987) Kharkov: UkrNIINTI 1987

[14] Mohar, B.: On the minimal genus of 2-complexes. J. Graph Theory 24, 281-290 (1997)

[15] Mohar, B.: Personal communication to S. Lawrencenko (31 Mar 2009)

[16] Negami, S.: A verbal comment on a talk given by S. Lawrencenko at 20th (International) Workshop on Topological Graph Theory in Yokohama (27 Nov 2008)

[17] Pach, J., Tóth, G.: Crossing number of toroidal graphs. In: M. Klazar (ed.) et al.: Topics in discrete mathematics, Dedicated to Jarik Nešetřil on the occasion of his 60th birthday (Algorithms and Combinatorics, vol. 26, pp. 581-590) Berlin: Springer 2006

[18] Ringeisen, R.D.: Graphs of given genus and arbitrarily large maximum genus. Discrete Math. 6, 169-174 (1973)

[19] Ringel, G., Youngs, J.W.T.: Solution of the Heawood map-coloring problem. Proc. Nat. Acad. Sci. U.S.A. 60, 438-445 (1968)

[20] Steinitz, E., Rademacher, H.: Vorlesungen über die Theorie der Polyeder. Berlin: Springer 1934