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Three nonisomorphic triangulations of an orientable surface with the same complete graph

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Abstract

We identify three mutually nonisomorphic triangulations of the closed orientable surface of genus 20, each with the complete graph on 19 vertices.

The following problem is basic in the theory of graph re-embeddings (a branch of topological graph theory [8]): Examine the number of nonisomorphic embeddings of a given graph G in an orientable, $S=S_g$, or a nonorientable, $S=\tilde{S}_g$, (closed) surface S of genus g. Two embeddings $f_i: G \to S$ $(i=1,2)$ are regarded as isomorphic provided there exist a homeomorphism h of S and an automorphism α of G so that $f_2(G)=h(f_1(\alpha(G)))$. In case G triangulates S, it follows from Euler's formula that any embedding of G in S is a *triangulation*, that is, an embedding each face of which is bounded by a cycle consisting of three edges of G. Combinatorially, triangulations T_1 and T_2 of S, with the same graph, are isomorphic if and only if there exists a bijection $\theta: V(T_1) \to V(T_2)$, called an *isomorphism*, between their vertex sets so that a cycle u–v–w–u bounds a face of T_1 if and only if $\theta(u)$–$\theta(v)$–$\theta(w)$–$\theta(u)$ bounds one of T_2 (not necessarily with the same orientation).

Presently, this problem is far from being fully solved even in the case of $G=K_n$, the complete graph on n vertices, and even with the additional restriction that G triangulates S. Let $\#\mathrm{tri}(G,S)$ be the number of nonisomorphic triangulations of S with

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a graph G. Denote by $\gamma(G)$ [resp., $\tilde{\gamma}(G)$] the *genus* [resp., *nonorientable genus*] of G, that is, the minimum g such that G embeds in S_g [resp., $\tilde{S}_g$]. The formulas for $\gamma(K_n)$ and $\tilde{\gamma}(K_n)$, $n \geqslant 3$, are known [6, 5]:

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil,$$

$$\tilde{\gamma}(K_7) = 3,$$

$$\tilde{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil, \quad n \neq 7.$$

(Here we assume by agreement that $\tilde{S}_0 = S_0$.) Now, using Euler's formula, one may verify (see [8]) that $\#\mathrm{tri}(K_n, S_{\gamma(K_n)}) \neq 0$ if and only if $(n-3)(n-4)/12$ is an integer, and similarly in the nonorientable case with the one exception $n = 7$. Clearly, $\#\mathrm{tri}(K_3, S_0) = 1$ and $\#\mathrm{tri}(K_4, S_0) = 1$; furthermore, from the result of [2] (resp. of [3] or [4]), it follows that $\#\mathrm{tri}(K_6, \tilde{S}_1) = 1$ (resp. $\#\mathrm{tri}(K_7, S_1) = 1$). For $n > 7$ it is an open problem to determine $\#\mathrm{tri}(K_n, S_{\gamma(K_n)})$ [or $\#\mathrm{tri}(K_n, \tilde{S}_{\tilde{\gamma}(K_n)})$] unless the value is zero.

Although examples of nonisomorphic triangulations of a nonorientable surface with the same complete graph have been constructed (Arocha et al. [1]), none was known in the orientable case. In this note we fill this gap; we will now show that $\#\mathrm{tri}(K_{19}, S_{20}) \geqslant 3$ by exposing three mutually nonisomorphic triangulations $A, B, C: K_{19} \to S_{20}$. It is well known (see [8]) that any embedding of a given graph G in an orientable surface can be defined by the corresponding *rotation scheme* that specifies, for each vertex v of G, the *rotation* at v, that is, the cyclic order in which the adjacent vertices are placed around v. Associating the vertices of K_{19} with the elements of Z_{19}, the cyclic group of integers modulo 19, we define A, B, and C by the following rotation schemes:

A) 0: 1-6-2-12-16-5-18-10-17-4-7-9-8-3-15-13-14-11,

B) 0: 1-5-16-10-17-4-18-11-14-15-13-3-8-7-9-6-2-12,

C) 0: 1-6-15-12-11-9-13-14-16-4-10-2-5-18-7-3-17-8.

Here we only give the rotation at vertex 0; the rotation at vertex i $(1 \leqslant i \leqslant 18)$ of A, B, or C is obtained by adding i modulo 19 to each entry of the corresponding row above. The reader may verify that the so-defined embeddings are indeed triangulations.

We have checked with computer that A, B, and C are mutually nonisomorphic. To check whether two given triangulations T_1 and T_2 with the graph K_n are isomorphic or nonisomorphic, we do not need to examine all $n!$ bijections $\theta: V(T_1) \to V(T_2)$. For, fix two vertices — v_0 and v_1 — in $V(T_1)$; then, once $\theta(v_0)$ is fixed in $V(T_2)$, there are $n-1$ choices for $\theta(v_1)$; further, with $\theta(v_1)$ also fixed in $V(T_2)$, the images of the remaining vertices of T_1 are determined uniquely up to reversing the rotation at $\theta(v_0)$. Thus, instead of $n!$ candidates for isomorphism, it suffices to check only $2n(n-1)$ ones.

A triangulation T is said to be *tight* provided for any partition of $V(T)$ into three parts there exists a face of T having an incident vertex in each part. In [1], pairs of a tight and a nontight triangulations of the same nonorientable surface are constructed with the same complete graph, and a question is raised whether there exists a nontight triangulation of an orientable surface with some complete graph. It turned out that all the triangulations A, B, and C are tight (which we checked with computer), so that we had to establish their mutual nonisomorphy directly.

However, we observed that the faces of C can be partitioned into two parts so that neighboring faces are in different parts, which is not the case for A or B. In fact, C can be generated by using $K_{3,3}$ as an index one current graph, whereas both A and B arise from the cartesian product $K_2 \times K_3$ as an index one current graph (with differing current assignments). In each of the three cases the covering embedding of K_{19} in S_{20} determines a (19, 114, 18, 3, 2)-BIBD, but only for the bichromatic-dual case C does this split into two (19, 57, 9, 3, 1)-BIBDs, or Steiner triple systems; see [7].

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