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Structural characterization of projective flexibility [1]

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Abstract

A triangulation T of a fixed surface Σ is called flexible if its graph $G(T)$ has two or more labeled embeddings in Σ. We establish a structural characterization of flexible triangulations of the projective plane. © 1998 Elsevier Science B.V. All rights reserved

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By $V(\cdot)$ and $F(\cdot)$ we denote the sets of vertices and faces, respectively. A *3-cycle* is a cycle of length 3. A *triangulation* $T: G \to \Sigma$ of a fixed closed surface Σ with a graph G is an embedding of G in Σ with every face bounded by a 3-cycle of G. Combinatorially, triangulation T is completely determined by its graph $G = G(T)$ together with the face set $F(T)$. Two triangulations $T, T': G \to \Sigma$ are called *isomorphic* provided there is an *isomorphism* $\pi: T \to T'$, i.e., a permutation of $V(G)$ such that $uvw \in F(T)$ if and only if $\pi(u)\pi(v)\pi(w) \in F(T')$. Especially, an isomorphism $T \to T$ is called an *automorphism* of T. We say T and T' are *equivalent*, written $T = T'$, if $F(T) = F(T')$, i.e., any 3-cycle of G bounds a face either in both T and T' (not necessarily with the same orientation) or in neither; otherwise we say T and T' are *distinct*. Note that distinct [respectively, nonisomorphic] triangulations are distinguishable in the vertex-labeled [vertex-unlabeled] sense and that distinct triangulations may be isomorphic. An important theorem of Whitney [6] implies that any triangulation T of the 2-sphere Σ_0 is combinatorially unique, i.e., $F(T)$ is uniquely determined by $G(T)$. For higher-genus surfaces, Whitney's theorem fails and there may exist two or

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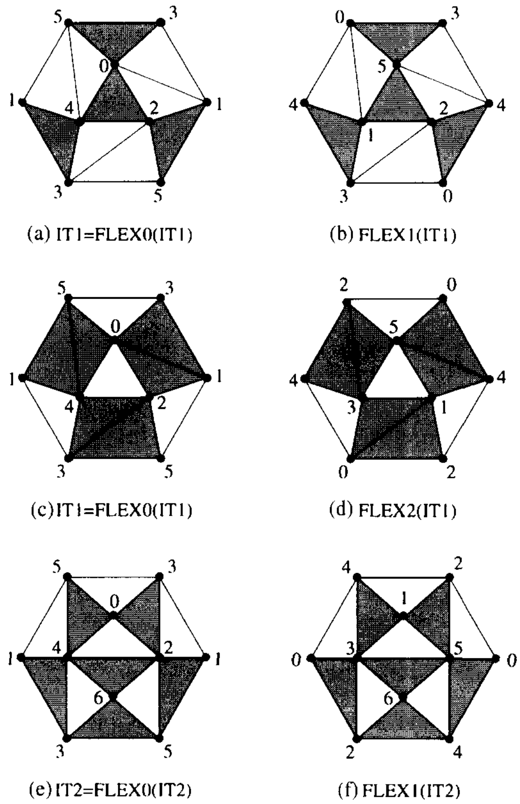

Fig. 1. Triangulations of the projective plane.

more distinct triangulations $G(T) \to \Sigma$ with the graph $G(T)$ of a given triangulation T of $\Sigma \neq \Sigma_0$; triangulation T will be fixed hereafter. In this note, we continue the development of our approach to graph re-embedding theory, which was begun in [3,4].

Denote by $\{\mathrm{FLEX}_i(T)\}$ the set of the *flexes* of triangulation T in surface Σ, i.e., the set of pairwise distinct (labeled) embeddings $G(T) \to \Sigma$ (necessarily triangulations). As matter of notation, we always assume that $\mathrm{FLEX}_0(T)$ is triangulation T itself, regarding it as *trivial flex*. A face of T is called *rigid*, with respect to Σ, if it is a face of each flex of T in Σ, and is called *flexible* otherwise. Each boundary edge of a flexible face occurs in more than two 3-cycles of $G(T)$ because every edge of any triangulation of Σ must occur in the boundaries of two faces. Triangulation T is called *rigid* if all its faces are rigid; otherwise T is called *flexible*. Clearly, T is flexible if and only if its graph $G(T)$ has two or more labeled embeddings in Σ, i.e., $|\{\mathrm{FLEX}_i(T)\}| \geqslant 2$.

We depict the projective plane $\tilde{\Sigma}_1$ as a regular hexagon with antipodal points on the boundary treated as identical. Then Fig. 1 presents some triangulations of $\tilde{\Sigma}_1$. The triangulations in each pair, (a, b), (c, d), and (e, f), are isomorphic but distinct,

Table 1
Flexes of the irreducible triangulations of the projective plane

$\mathrm{FLEX}_0(IT_1)$:	0 1 2 3 4 5	$\mathrm{FLEX}_0(IT_2)$:	0 1 2 3 4 5 6
$\mathrm{FLEX}_1(IT_1)$:	5 4 2 3 1 0	$\mathrm{FLEX}_1(IT_2)$:	1 0 5 2 3 4 6
$\mathrm{FLEX}_2(IT_1)$:	5 4 1 0 3 2	$\mathrm{FLEX}_2(IT_2)$:	1 0 2 3 4 5 6
$\mathrm{FLEX}_3(IT_1)$:	1 5 2 3 4 0	$\mathrm{FLEX}_3(IT_2)$:	1 6 2 3 4 5 0
$\mathrm{FLEX}_4(IT_1)$:	5 0 2 3 4 1	$\mathrm{FLEX}_4(IT_2)$:	0 6 2 3 4 5 1
$\mathrm{FLEX}_5(IT_1)$:	5 1 2 3 4 0	$\mathrm{FLEX}_5(IT_2)$:	6 1 2 3 4 5 0
$\mathrm{FLEX}_6(IT_1)$:	0 4 2 3 1 5		
$\mathrm{FLEX}_7(IT_1)$:	0 5 2 3 4 1		
$\mathrm{FLEX}_8(IT_1)$:	4 5 2 3 1 0		
$\mathrm{FLEX}_9(IT_1)$:	4 0 2 3 1 5		
$\mathrm{FLEX}_{10}(IT_1)$:	1 4 2 3 5 0		
$\mathrm{FLEX}_{11}(IT_1)$:	4 1 2 3 5 0		

since their face sets are distinct: they have only the shaded faces in common (respectively).

The operation of *shrinking* an edge v_1v_2 of T, denoted by $\mathrm{sh}\rangle v_1v_2\langle$, consists of collapsing this edge to a single vertex, v, and the two incident faces, v_1v_2u and v_1v_2w, to two edges: vu and vw, respectively. The inverse of this operation is called the *splitting*, $\mathrm{sp}\langle u,v,w\rangle$, of the *corner* $\langle u,v,w\rangle$, i.e., the pair of edges $\{vu,vw\}$. Recall that each edge of T occurs in the boundaries of exactly two faces; if an edge occurs in more than two 3-cycles of $G(T)$, it is called *unshrinkable*. We say T is an *irreducible triangulation* (of $\Sigma \neq \Sigma_0$) if each edge of T is unshrinkable; none of its edges can be shrunk further. Clearly, the whole family of triangulations of Σ can be obtained from the irreducible ones by repeatedly applying the operation of splitting. We shall use the complete list [1], up to isomorphisms, of irreducible triangulations of the projective plane $\tilde{\Sigma}_1$. This list consists of two members, namely IT_1 and IT_2 depicted in Figs. 1(a) (or (c)) and (e), respectively.

All flexes of IT_1 and IT_2 in the projective plane are presented in Table 1 (from [4]). The trivial flexes $\mathrm{FLEX}_0(IT_1) = IT_1$ and $\mathrm{FLEX}_0(IT_2) = IT_2$ are depicted in Figs. 1(a) (or (c)) and (e), respectively. Every line in Table 1 is a permutation of the first line. To obtain a picture of $\mathrm{FLEX}_i(IT_1)$, merely replace the labels in Fig. 1(a) as the ith permutation prescribes; and similarly for $\mathrm{FLEX}_i(IT_2)$. For instance, to obtain $\mathrm{FLEX}_1(IT_1)$, replace the labels 0, 1, 2, 3, 4, 5 in Fig. 1(a) with 5, 4, 2, 3, 1, 0, respectively; see Fig. 1(b). Similarly, $\mathrm{FLEX}_2(IT_1)$ and $\mathrm{FLEX}_1(IT_2)$ are shown in Figs. 1(d) and (f), respectively. To understand the proof of Theorem 3 below, it is helpful to draw pictures of the twelve flexes of IT_1 and the six flexes of IT_2.

The flex set $\{\mathrm{FLEX}_i(T)\}$ evolves under splittings of T; some of the flexes survive and some are destroyed. Let $T' = \mathrm{sp}\langle u,v,w\rangle(T)$ and let v^+, v^- denote the two images of vertex v under the splitting; note that the two faces v^+v^-u and v^+v^-w are certainly rigid. Clearly, for each j, there is a unique i such that:

$$\mathrm{sh}\rangle v^+v^-\langle(\mathrm{FLEX}_j(T')) = \mathrm{FLEX}_i(T). \qquad (1)$$

We say that $\text{FLEX}_i(T)$ *survives* under the splitting $\text{sp}\langle u,v,w\rangle$ provided Eq.(1) holds for some j. Two corners $\langle u,v,w\rangle$ and $\langle x,v,y\rangle$ are said to *cross* each other (at vertex v) in triangulation T if there is a homeomorphism of star(v, T) onto the unit disk in the complex plane such that the image of $\langle u,v,w\rangle$ follows the real axis and the image of $\langle x,v,y\rangle$ follows the imaginary axis.

Lemma 1 (Mechanism of evolution [3,4]). *For $uvw \in F(T)$, $\text{FLEX}_i(T)$ survives under $\text{sp}\langle u,v,w\rangle : T \mapsto T'$ if and only if $uvw \in F(\text{FLEX}_i(T))$. When $uvw \notin F(T)$, $\text{FLEX}_i(T)$ survives if and only if it has no face vxy such that the corners $\langle x,v,y\rangle$ and $\langle u,v,w\rangle$ cross each other in T.*

Corollary 2. *There is a constant upper bound on the number of flexible faces in a triangulation of a fixed surface Σ.*

Proof. It is clear that the two new faces produced by a splitting are always rigid and that splitting preserves the rigidity of a face. Hence, the number of flexible faces cannot increase by splitting. On the other hand, it is known [2] that there are at most finitely many irreducible triangulations of any surface Σ, and the result follows. □

Three important simplicial 2-complexes are determined by the faces shaded in Fig. 1, namely: the *bunch of four triangles*, BT, shaded in Fig. 1(a); the *bunch of three squares*, BS, shaded in Fig. 1(c); and the *bunch of three bouquets*, BB, shaded in Fig. 1(e). Observe that the triangulation IT_2 can be obtained from the triangulation IT_1 by retriangulating the underlying space $|BS|$ of BS, more precisely, IT_2 contains the following three "squares:" 2435, 1405 and 1203 (Fig. 1(e)).

Theorem 3. *All flexible triangulations of the projective plane $\tilde{\Sigma}_1$, up to isomorphisms, can be generated from the triangulations IT_1 (Figs. 1(a) and (c)) and IT_2 (Fig. 1(e)) by retriangulating the underlying space of one of the bunches, BT, BS, or BB, without adding new vertices to their boundaries and without producing multiple edges.*

Proof. Observe first that the 'common parts' of the pairs of distinct triangulations (a, b), (c, d), and (e, f) of Fig. 1 are exactly the bunches BT, BS, and BB, respectively. It follows that any triangulation obtained by retriangulating $|BT|$ or $|BS|$ in IT_1, or $|BB|$ in IT_2, is indeed flexible. Therefore, our job is to prove that *any* flexible triangulation can be obtained in this fashion. Recall that each triangulation of $\tilde{\Sigma}_1$ can be obtained from IT_1 or IT_2 by a sequence of splittings. To characterize the splittings of the sequence under which the resulting triangulation is still flexible, we shall examine the evolution of the sets $\{\text{FLEX}_i(IT)\}$ more delicately, for the triangulations $IT \in \{IT_1, IT_2\}$. The purpose of our next steps is to come to the following conclusion: once a splitting of the sequence affects some two neighboring faces of IT, a whole copy of the bunch BS is automatically fixed in IT, but the triangulation is still flexible; furthermore, the next splittings of the sequence will either retriangulate the underlying space of the bunch BS fixed or result in a rigid triangulation.

To *fix a face (deliberately)* in IT means to make that face rigid by merely deleting the flexes of IT which do not contain it. On the other side, fixing a face may turn some other face xyz into a rigid face, which is the case when all the flexes remaining after the deletion contain xyz; we say, then, that face xyz is *fixed automatically*. It is clear from the mechanism of evolution that fixing a face of IT is equivalent (from the rigidity–flexibility viewpoint) to splitting one of its actual corners followed by repeatedly splitting the corners inside that (ex-)face; this process retriangulates the interior of that face (without adding more vertices to its boundary) and preserves its rigidity. Similarly, in the case of two neighboring faces, fixing them is equivalent to retriangulating the interior of their union, which again can be done by repeatedly splitting appropriate corners. For the sake of simplicity, the reader may imagine that a single vertex is placed in every face which is fixed (deliberately or automatically), the 'center' of the face, and joined to each vertex in its boundary.

We want to generate all flexible triangulations up to isomorphisms. So, to verify the statements below, it will be helpful to bear in mind that the automorphism group of IT_1 is flag-transitive; see [3,4]. (Recall that a *flag* of a triangulation is an incident vertex-edge-face triple.) Also, each vertex of degree four [respectively, six] of IT_2 can be sent to vertex 6 [vertex 2] by an appropriate automorphism of IT_2; furthermore, the stabilizer of vertex 6 [vertex 2] acts transitively on the set of edges $\{62, 64, 63, 65\}$ [on the sets $\{20, 26, 21\}$ and $\{23, 24, 25\}$].

Using Lemma 1 and Table 1, the reader may verify that if we fix any collection of pairwise edge-disjoint faces in IT, we will have a triangulation which is still flexible and can be obtained either from IT_1 by fixing appropriate faces in a copy of the bunch BT or in a copy of the possibly retriangulated bunch BS, or from IT_2 by fixing some faces in a copy of the bunch BB. We may need to retriangulate $|BS|$ in the case of $IT = IT_2$; for instance, if we fix faces 054, 032, 634, and 652 in IT_2 (Fig. 1(e)). Furthermore, if we fix any pair of neighboring faces in IT, we will always have a flexible triangulation which can be obtained from IT_1 by fixing a whole bunch BS, i.e., all the faces of a possibly retriangulated copy of BS.

Assume that v is one of the original vertices of IT and that u_i $(i = 1, 2)$ is one of the original vertices or the center of one of the original faces of IT. Then the reader may verify that if the edges vu_1 and vu_2 are in nonneighboring faces, then the triangulation $\mathrm{sp}\langle u_1, v, u_2\rangle(IT)$ is rigid unless v is a vertex of degree four in IT_2, say vertex 6, in which event $\mathrm{sp}\langle u_1, v, u_2\rangle$ fixes the whole bunch BS consisting of retriangulated 'squares' 2435, 1405, and 1203 (Fig. 1(e)).

Under fixing the whole bunch BT shaded in IT_1 (Fig. 1(a)), only two flexes survive, namely: $\mathrm{FLEX}_0(IT_1)$ and $\mathrm{FLEX}_1(IT_1)$ (Figs. 1(a) and (b)). Furthermore, fixing the whole bunch BS shaded in IT_1 (Fig. 1(c)) destroys all the flexes except $\mathrm{FLEX}_0(IT_1)$ and $\mathrm{FLEX}_2(IT_1)$ (Figs. 1(c) and (d)). Similarly, after fixing the whole bunch BB shaded in IT_2 (Fig. 1(e)), only $\mathrm{FLEX}_0(IT_2)$ and $\mathrm{FLEX}_1(IT_2)$ are left (Figs. 1(e) and (f)). Furthermore, with B designating any one of the three bunches fixed, it is routine to verify that any splitting $\mathrm{sp}\langle u_1, v, u_2\rangle$ such that $\langle u_1, v, u_2\rangle \not\subset |B|$ makes the triangulation rigid. Now the result is obvious. □

Since fixing a single face [respectively, fixing a whole bunch BS] in IT_1 reduces the size of its flex set to 6 [to 2], we are led to the following result (a similar result, for maximum connectivity $\kappa = 5$, is derived by Negami via a different method [5]).

Corollary 4 (Lawrencenko [3]). *Let T be an arbitrary triangulation of the projective plane, not IT_1 or IT_2. Then, if connectivity $\kappa(G(T)) = 3$, 4, or 5, we have $|\{\mathrm{FLEX}_i(T)\}| \leqslant 6$, 2, or 1, respectively. Furthermore, for each κ, equality holds on infinitely many triangulations.*

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