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Constructing the Graphs That Triangulate Both the Torus and the Klein Bottle

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We show how to construct all the graphs that can be embedded on both the torus and the Klein bottle as their triangulations. © 1999 Academic Press

1. INTRODUCTION

Throughout this paper we assume that the term "graph" disallows loops and multiple edges. Whenever we say "surface," we shall mean a closed, compact and connected 2-manifold; it is well-known that each orientable surface is a sphere with $k \geqslant 0$ handles, denoted by S_k, while each nonorientable one is a sphere with $k > 0$ crosscaps, denoted by $\tilde{S}_k$. In general we follow the terminology and notation of [17].

An embedding T of a graph G in a surface S is called a *triangulation* of S if each face of T is bounded by a 3-*cycle* of G, that is, a cycle consisting of three edges of G. If one embedding of G in some surface is a triangulation, then it follows from Euler's formula that all embeddings of G in surfaces of the same Euler characteristic are triangulations.

It follows from Ringel's results that if $\chi = (7n - n^2)/6$ is a negative even integer, then the complete graph K_n triangulates both $S_{1-\chi/2}$ and $\tilde{S}_{2-\chi}$; see [15, 16]. Such is not the case when $n = 7$ (so that $\chi = 0$); it is well-known

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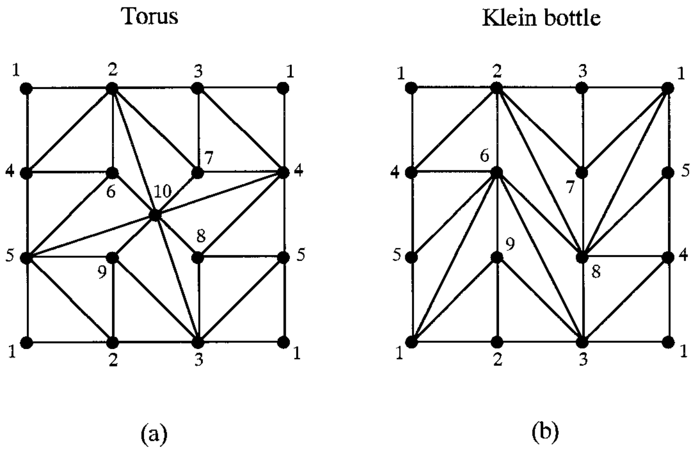

FIG. 1. Triangulations of a torus and a Klein bottle.

that K_7 triangulates the torus S_1, but does not embed in the Klein bottle $\tilde{S}_2$ (see Franklin [6]). Curiously, while S_1 and $\tilde{S}_2$ have the same Euler characteristic $\chi = 0$, some graphs triangulate S_1 but not $\tilde{S}_2$, some $\tilde{S}_2$ but not S_1, and some both S_1 and $\tilde{S}_2$. The following are some more examples to illustrate this phenomenon.

Figure 1 presents triangulations (a) of S_1 and (b) of $\tilde{S}_2$; the torus and the Klein bottle are depicted as a square with opposite sides identified in pairs as the labels prescribe. Let $G(n, n, n)$ denote the graph obtained from the complete bipartite graph $K_{n,n}$ by removing n mutually disjoint edges. It is easy to see that the graph of (a) has as a subgraph $G(5, 5, 5)$ obtained from the graph $K_{5,5}$ having as parts the vertex subsets $\{1, 6, 7, 8, 9\}$ and $\{10, 3, 5, 2, 4\}$. The graph $G(5, 5, 5)$ does not embed in $\tilde{S}_2$ (see Mohar [12]), so that the graph of (a) does not either. On the other hand, we shall see in Section 2 that the graph of (b) does not embed in S_1.

Figure 2 presents triangulations (a) of S_1 and (b) of $\tilde{S}_2$, which we will denote by T_L and K_L, respectively, both with the graph $L = (P_4 + \bar{K}_2) + K_2$. The labels are such that, in constructing L, $(6, 8, 5, 1)$ is the path P_4, $\{3, 7\}$ is the vertex set of $\bar{K}_2$, and $\{2, 4\}$ of K_2. The shaded faces are those bounded by the same (labeled) 3-cycles of L in both T_L and K_L.

Let us subdivide the shaded faces in T_L and K_L, in the same way in both, with the only restriction that this process must produce triangulations. In particular, the edge 67 may be subdivided into a path, but the other edges on the boundary of each shaded face must not be subdivided. Obviously, by doing so, we can obtain infinitely many pairs of triangulations of S_1 and $\tilde{S}_2$ each pair of which have the same graph. The main result of this paper is that every such pair can be obtained in this way:

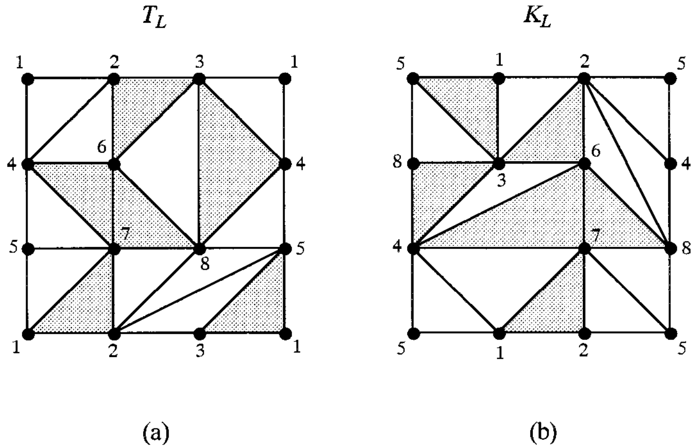

FIG. 2. Triangulations of a torus and a Klein bottle with the same graph L.

THEOREM 1. *Given a pair of triangulations of the torus and the Klein bottle, both with the same graph, such pair can be obtained from the pair* $\{T_L, K_L\}$, *presented in Fig. 2, by appropriately triangulating the shaded faces.*

We shall prove this theorem in Section 4, suggesting a procedure for generating all such pairs from $\{T_L, K_L\}$ by successively splitting the corresponding vertices in both T_L and K_L so as to subdivide them only within the shaded faces. Such recursive constructions of many families of planar graph embeddings are known while for graphs in other surfaces much less is known. This topic received increasing attention during the past decade; see, for example, [1–4, 8–14].

Theorem 1 immediately implies the following:

COROLLARY 2. *There is precisely one* 5-*connected graph, L, which triangulates both the torus and the Klein bottle, and there is none which is* 6-*connected.*

2. IRREDUCIBLE TRIANGULATIONS

Let $T(G)$ be a triangulation of some surface with a graph G. By *shrinking* an edge v^+v^- in $T(G)$ we mean that this edge shrinks into a single vertex v and the two faces, v^+v^-x and v^+v^-y, meeting this edge degenerate into two edges vx and vy, respectively. The inverse of this operation is called *splitting* the vertex v along the edges vx and vy. We shall say that an edge is *shrinkable* provided that shrinking the edge produces an embedding of a (simple) graph. If no edge of $T(G)$ is shrinkable we say that the triangulation is *irreducible.*

The only impediment to shrinkability is the creation of multiple edges. This would happen only when the edge appears in more than two of the 3-cycles of G, so that the shrinkable edges of $T(G)$ are those appearing in exactly two 3-cycles each (of course, those two bound faces of $T(G)$). Therefore, rather than being a property of $T(G)$, being irreducible is a property of its graph G. In particular, any embedding of K_7 in S_1 is necessarily an irreducible triangulation.

It has been already shown, in [5, 7, 11, 13, 14], that the set $\mathscr{I}(S)$ of irreducible triangulations of any surface S is finite (up to isomorphisms, see Section 3). In particular, Lawrencenko [8] has determined $|\mathscr{I}(S_1)| = 21$, identifying all the members of $\mathscr{I}(S_1)$ explicitly; two of them are presented in Figs. 1a and 2a. Recently, Lawrencenko and Negami [9] have classified all the members of $\mathscr{I}(\tilde{S}_2)$, which are 25 in number. Figures 1b and 2b present two of the members of $\mathscr{I}(\tilde{S}_2)$.

Let us now prove what we asserted in the introduction—that the graph of Fig. 1b does not embed in S_1. Actually, if it did, then it would occur in $\mathscr{I}(S_1)$ since each of its edges appears in more than two 3-cycles, contrary to [8].

The following fact is one of the key facts used to prove Theorem 1:

LEMMA 3. *There exists precisely one graph, L, which admits irreducible triangulations of both the torus and the Klein bottle.* ∎

This has been shown as Theorem 10 in [9], where the authors have discussed on some partial structures of irreducible triangulations on the Klein bottle $\tilde{S}_2$ which make them not embeddable in the torus. In their classification, the unique member of $\mathscr{I}(\tilde{S}_2)$ which L admits is denoted by Khl and is isomorphic to K_L given in Fig. 2b. (Before the authors established the complete list of irreducible triangulations of the Klein bottle, they also had checked that the member of $\mathscr{I}(S_1)$ embeddable in the Klein bottle is isomorphic to T_L given in Fig. 2a, using a computer program and already concluded Theorem 1 in 1994. Now we can give a computer-free proof with the above lemma.)

3. ISOMORPHY OF PAIRS OF TRIANGULATIONS

A triangulation T of some surface S with a given graph G is assumed to be well-defined by the set $F(T)$ of the 3-cycles of G bounding faces of T; thus we only distinguish triangulations $G \to S$ up to homeomorphisms of S.

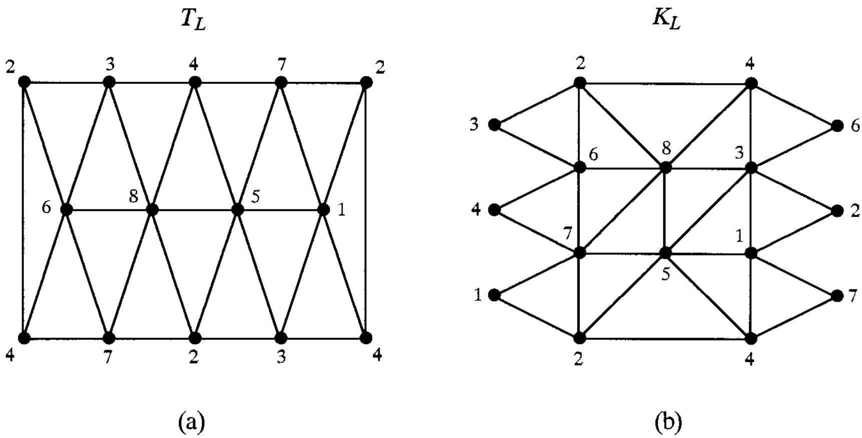

FIG. 3. Symmetric appearances of T_L and K_L.

By $\mathrm{Aut}(G)$ we denote the automorphism group of G. For $\theta \in \mathrm{Aut}(G)$ we denote by $\theta(T)$ the triangulation, with the same graph G, defined by:

$$F(\theta(T)) = \{\theta(u)\,\theta(v)\,\theta(w) \mid uvw \in F(T)\}.$$

Let T and T' be two triangulations, both with a graph G. We shall write $T = T'$ if $F(T) = F(T')$. If there exists $\theta \in \mathrm{Aut}(G)$ with $\theta(T) = T'$, then T and T' are said to be *isomorphic*. In particular, if $\theta(T) = T$, then such an automorphism $\theta \in \mathrm{Aut}(G)$ of G is called a *symmetry* of T. The set of symmetries of T is called the *symmetry group* of T and is denoted by $\mathrm{Sym}(T)$. It is clear that $\mathrm{Sym}(T)$ is a subgroup of $\mathrm{Aut}(G)$ and that if $\mathrm{Sym}(T)$ has index n in $\mathrm{Aut}(G)$, then G admits precisely n triangulations which are isomorphic to T but are all distinct.

Two pairs $\{T_1, T_2\}$ and $\{T'_1, T'_2\}$ of triangulations of some surfaces with the same graph G are said to be isomorphic pairs if there exists $\theta \in \mathrm{Aut}(G)$ such that $\theta(T_1) = T'_1$ and $\theta(T_2) = T'_2$.

LEMMA 4. *Each pair of triangulations, both with the graph L, of the torus and the Klein bottle is isomorphic to the pair $\{T_L, K_L\}$.*

Proof. First we should identify the symmetry groups $\mathrm{Sym}(T_L)$ and $\mathrm{Sym}(K_L)$. Since $L = (P_4 + \bar{K}_2) + K_2$, its automorphism group $\mathrm{Aut}(L)$ is an abelian group of order 8 generated by (16)(58), (37), and (24):

$$\mathrm{Aut}(L) = \langle (16)(58), (37), (24) \rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2.$$

TABLE I

Automorphisms of the Graph L

θ_1	θ_2	θ
id	id	id
id	(24)	(37)(16)(58)
id	(37)	(24)(16)(58)
id	(24)(37)	(24)(37)
(24)	id	(24)(37)(16)(58)
(24)	(24)	(24)
(24)	(37)	(37)
(24)	(24)(37)	(16)(58)

It is routine to determine $\mathrm{Sym}(T_L)$ and $\mathrm{Sym}(K_L)$. The symmetric forms of T_L and K_L presented in Fig. 3, however, suggest that

$$\mathrm{Sym}(T_L) = \{\mathrm{id}, (24)(37), (37)(16)(58), (24)(16)(58)\};$$
$$\mathrm{Sym}(K_L) = \{\mathrm{id}, (24)(37)(16)(58)\}.$$

Thus, $\mathrm{Aut}(L)$ has the following coset decompositions:

$$\begin{aligned}\mathrm{Aut}(L) &= \mathrm{Sym}(T_L) \cup (24)\ \mathrm{Sym}(T_L)\\ &= \mathrm{Sym}(K_L) \cup (24)\ \mathrm{Sym}(K_L) \cup (37)\ \mathrm{Sym}(K_L) \cup (24)(37)\ \mathrm{Sym}(K_L).\end{aligned}$$

This implies that L admits 2 distinct triangulations on the torus and 4 on the Klein bottle, corresponding to these cosets:

$$T_L, (24)(T_L); \qquad K_L, (24)(K_L), (37)(K_L), (24)(37)(K_L).$$

Therefore, any pair of triangulations $\{T_1, T_2\}$ with the graph L of the torus and the Klein bottle is isomorphic to $\{\theta_1(T_L), \theta_2(K_L)\}$ for suitable automorphisms $\theta_1, \theta_2 \in \mathrm{Aut}(L)$. Table I specifies $\theta \in \mathrm{Aut}(L)$ corresponding to each possible pair $\{\theta_1, \theta_2\}$. For every such pair, it is routine to verify that $\theta(T_L) = \theta_1(T_L)$ and $\theta(K_L) = \theta_2(K_L)$. Thus, $\{T_1, T_2\}$ is isomorphic to $\{T_L, K_L\}$. ∎

4. COMMON SPLITTINGS

As usual, denote by $V(G)$ and $E(G)$ the vertex and the edge sets of a graph G, respectively. Let T_i $(i = 1, 2)$ be two triangulations of some surfaces, both with a graph G. The *link* of a vertex $v \in V(G)$ in T_i is a cycle of G induced by the set of edges $\{uw \in E(G) \mid vuw \in F(T_i)\}$ and is denoted by link(v, T_i). Let vx, $vy \in E(G)$ with $x \neq y$. Then the vertices x and y divide link(v, T_i) into two open paths, A_i and B_i. If $V(A_1) = V(A_2)$ (or, equivalently, $V(B_1) = V(B_2)$), splitting v along vx and vy is said to be a *common splitting* for T_1 and T_2. Note that applying a common splitting to T_1 and T_2 produces a pair of triangulations with the same graph again.

Proof of Theorem 1. Given two triangulations T_1 of S_1 and T_2 of $\tilde{S}_2$ with the same graph, by repeatedly shrinking the corresponding shrinkable edges simultaneously in T_1 and T_2, we shall finally obtain a pair of irreducible triangulations of S_1 and $\tilde{S}_2$, still both with the same graph. By Lemmas 3 and 4, this pair is isomorphic to $\{T_L, K_L\}$. Thus, by the reverse sequence of splittings applied to $\{T_L, K_L\}$, we return to the pair $\{T_1, T_2\}$. Clearly all these splittings are common for the corresponding pairs of triangulations. One can easily verify that a splitting is common for T_L and K_L if and only if it is done along the edges 64 and 68, or 74 and 78, or vx and vy, where vxy is one of the shaded faces; see Fig. 2; it follows that any sequence consisting of only common splittings, applied to $\{T_L, K_L\}$, can be done by accordingly changing T_L and K_L only within the shaded faces. This completes the proof of Theorem 1. ▮

Finally, we note that from the pair $\{T_L, K_L\}$ the whole family of such pairs of triangulations of S_1 and $\tilde{S}_2$ (both triangulations of each pair with the same graph) can be recursively generated with the only operation of common splitting.

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